Which Propositions Have Objective Probabilities?

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Abstract

It appears that the physical world contains facts regarding objective probability. What I will call Simple Anti-Humean Realism (SAR) treats these as primitive and fundamentally modal features of reality. I will argue that advocates of SAR face an interesting internal challenge about how to explain why certain propositions lack objective probabilities – which may also apply to other views.

1 Introduction

There appear to be facts about objective physical possibility. For example, modern physics famously appears to predict fundamentally probabilistic outcomes. Some philosophers attempt to reductively analyze claims about objective probability, by grounding them in facts about other things, like patterns within a Humean mosaic of actual events [12].

In contrast, the default naive realist perspective which I will call **Simple Anti-Humean Realism** (SAR) rejects such reductive analyses. It maintains that objective probability facts are not grounded in, or reducible to, other kinds of facts[1]. Instead, it asserts that fundamental reality includes primitive and irreducible modal facts about objective physical probabilities, just as some might postulate fundamental modal facts about logical, physical, or metaphysical possibility.

Because simple anti-Humean realism takes probability talk at face value, it can seem like the natural position to take absent some conflicting larger philosophical project (e.g., empiricism, materialism, ambitions for extreme conceptual parsimony). And perhaps SAR gains further support from the literature on apparent problems for more ambitiously reductive approaches to probability (like David Lewis' Humean analysis [12]).

[1] I take the view of fundamental stochastic laws described in Maudlin's 'Three Roads to Objective Probability'[16] to be an example of simple anti-humean realism.

In this paper I will argue that simple anti-Humean realists about objective probability face an interesting challenge about how to philosophically understand the fact that certain sets of metaphysically (and physically) possible outcomes lack probabilities.

In §2 I'll review how a classic thought experiment (involving physical hypotheses described using non-measurable sets) suggests some propositions can lack objective probabilities. In §3 I'll argue that accepting this appearance raises a prima facie puzzle and philosophical choice point for SAR advocates as follows. How could reality play favorites by treating some propositions as special in this way (i.e., in having objective probabilities while others lack them)? If it does, what explains why some propositions have probabilities while others lack them?

In §4, I'll highlight some snags for what might seem like an easy answer to the above challenge – saying propositions associated with non-Lebesgue measurable sets are somehow intrinsically unsuited to have objective probabilities. In §5 I'll discuss an alternative approach which makes facts about which propositions have objective probabilities world relative. I currently think this approach is promising, but I'll suggest that accepting it would leave us with significant philosophical work to do.

In §6, I'll conclude by reviewing possibilities for avoiding the explanatory demand by rejecting the explanandum (and hence rejecting some step in the conventional argument that propositions can lack objective probabilities), and explaining why I think prospects for doing this look grim[2].

Overall, my aim is not to argue for or against SAR. Rather I hope to highlight what I take to be an interesting puzzle and philosophical choice point which I think arises especially clearly for fans of SAR, but may also be relevant to other theories of objective physical possibility.

2 Probability Gaps

We seem to assign probabilities to something like coarsely individuated propositions (e.g., sets of possible worlds, or events qua sets of possible outcomes for an experiment) perhaps considered relative to a point in time[3]. For whenever

[2]In the appendix, I'll discuss an interesting contemporary proposal under this umbrella in more detail

[3]To motivate relativization of objective probability to (something like) a time, imagine a case with objective physical probabilities corresponding to the following. Three coins are flipped in sequence and all come up heads. We might say the following. At the time of the

two claims have the same possible worlds truth conditions – i.e., are true in the same set of metaphysically possible worlds – we expect them to have the same objective probability[4]. But do all coarsely individuated propositions (sets of possible worlds) have objective physical probabilities?

Naively you might expect that they should. For example, you might say something like the following. Vague claims which don't correspond to a precise division of the set of possible worlds could lack probability. But whenever a thought is definite enough to precisely divide the set of possible worlds into worlds where it is true and those where it is false, there should be an objective probability (relative to any given time) of that thought being true.

Additionally you might think there must be a fact about what subjective probability someone omniscient about the laws of nature and the history of the world up to a certain point (but ignorant about facts after this point etc.) should give to an arbitrary outcome P. And then you might say that the objective probability of P should correspond to this subjective probability.

Either way, the following thesis might look attractive.

Objective Probabilities for All Sets of Outcomes:

- For every proposition (qua set of possible worlds) P and time t, there's an objective probability 'at' time t, that P will wind up being true.
- For every set S of possible outcomes to an experiment, there's an objective probability (at the time of the commencement of the experiment) that an outcome in S occurs.

2.1 Motivating Probability Gaps

However there are powerful reasons for denying the above principles, as shown by the following classic thought experiment.

first coin being flipped, the objective probability of all flipped coins coming up heads is 1/8th. However when the second coin is flipped (since the first coin has already come up heads), even the objective probability that all coins will come up heads is 1/4.

In what follows I will sometimes talk about the probability of an event relative to a possible world and time. But I don't mean to take a position on exactly *what* objective physical probabilities should be regarded as relative to, as this question seems unrelated to the puzzle proposed in this paper.

[4]For example, we think that the objective probability (at launch) that a rocket will reach Hesperus must be the same as the objective physical probability that this rocket will reach Phosphorus. And we take the current objective probability that global temperatures will rise more than 3 degrees to be the same as the current objective physical probability that global temperatures will rise more than 3 degrees and all triangles are trilaterals.

Imagine a dart that is intuitively 'equally likely to land anywhere' on a rectangular dartboard, which spans the interval [-1,2] (with distances measured in meters from some physically specified zero point). Since the dart is equally likely to land in any regions with the same area, it should have probability 1/3rd of landing in the middle third of the dartboard, corresponding to the interval [0,1].

However, Vitali proved the following claim (from the standard ZFC axioms of set theory) [3]. There is a countably infinite list of sets of real numbers $V_1, V_2, V_3, \ldots$ (call these the Vitali sets), which are disjoint, contained in the region [-1,2], jointly include all points in [0,1] and have the following property.

- Each set of numbers V_i can be produced by shifting the first set of numbers V_1 over by some amount. That is, for each region V_i, there's a real number q_i such that $V_i = \{y | y = x + q_i \text{ for some x in } V_1\}$.

So, intuitively (by appeal to a *translation invariance* intuition that shifting a region over on the dartboard by q_i meters should not make a difference to the probability that our imagined fair dart lands there[5]), we would expect the dart to have equal probability of landing in the physical space corresponding to any one of the Vitali sets V_i. Now consider the following proposition.

> U: The dart will land somewhere in the union of the spatial regions corresponding to $V_1, V_2, ...$

What objective probability can U have? By countable additivity (one of the standard axioms of probability), P(U) should be equal to the sum of the probabilities that the dart lands in each each specific region V_i – since all of these regions are disjoint.

By the translation invariance point above, the probability of our dart landing in V_1 must be the same as it landing in V_i for any i. So the dart's probability $P(U)$ of landing in some such region should be either 0 (if the probability of it landing in V_1 is 0) or infinite (if the probability of landing on V_1 is > 0). The latter scenario is clearly impossible. So the probability of U, i.e. the dart landing in the union of the Vitali sets must be 0.

[5]One might try to express this intuition by saying that in the intuitively metaphysically possible informally described situation above, the probability of a dart landing in any region must be proportional to the area of that region, and areas are preserved by shifting all points in a region by a constant distance. But as we will discuss below, Vitali sets are have no Lebesgue measure and arguably physical regions corresponding to them should not be thought of as having areas.

But jointly the Vitali sets include every point in [0,1]. And we said the dart has probability $\frac{1}{3}$ of landing in this region[6]. So, by monotonicity (a consequence of the standard probability axioms), $P(U)$ is $\geq \frac{1}{3}$. Contradiction.

Accordingly, there seem to be fully conceivable and metaphysically possible situations where some precise propositions – like the claims about darts landing in Vitali regions above – would lack objective probabilities[7]. Thought experiments like this one, involving (physical possibilities somehow associated with) non-Lebesgue measurable sets[8], have motivated the following moral. A probability function correctly assigning objective probabilities to *some* sets of physically possible outcomes can't always be correctly extended to assign probabilities to *all* sets of physically possible outcomes.

Accordingly standard (Kolmogorov) approaches to probability don't require a probability function to assign values to all sets of outcomes considered. Instead they consider a probability triple consisting of the following

- a sample space, Ω, "which is the set of all possible outcomes for some random process or 'experiment' being modeled"
- an event space, which is "a set of events, F , an event being a set of outcomes in the sample space."
- and a probability function P which only must assign probabilities to all the events *in this event space* – rather than to all sets of possible outcomes (i.e., all subsets of the sample space Ω).

And a probability function can satisfy all the standard (Kolmogorov) probability axioms without assigning probabilities to all subsets of its sample space Ω. The event space doesn't need to be the powerset of the outcome space. It just needs to form a σ algebra for the set of possible outcomes[9].

A representative textbook presentation[10] of such basic probability theory

[6]For this region makes up a third of the total region corresponding to [-1,2], which is occupied by the dart board.

[7]I will survey some options for resisting this conclusion in §6

[8]The Vitali sets are influential examples of non- Lebesgue measurable sets. The Lebesgue measure on the reals is unique translation invariant measure assigning the unit interval measure 1 (while assigning some measure to all intervals [1]). And the Lebesgue measure on $\mathbb{R}^n$ straightforwardly generalizes this. An exact parallel to the reasoning above (but applied directly to sets of numbers not sets of spatial points) shows the Vitali sets cannot have a Lebesgue measure.

[9]That is, F just needs to be a subset of the $P(\Omega)$ which contains Ω and is closed under complement and countable union.

[10]Krishnapur's lecture notes 'Probability Theory - Measure theoretical framework' make the same motivating point as follows "By definition, we talk of probabilities only of measurable

motivates this permissiveness (only requiring probability functions to be defined on an event space rather than all all subsets of the set of outcomes) as follows.

> Not every subset of the sample space Ω must necessarily be considered an event: some of the subsets are simply not of interest, others cannot be 'measured'. [4]

As we will see below, philosophical questions arise about how to understand this suggestion that *sets of possible outcomes* 'can't be measured' and therefore cannot be assigned a probability (as opposed to, e.g., that a set of numbers lacks a *Lebesgue measure*)[11].

However, I take the practical stance adopted (only requiring probability functions to be defined on an event space not the full powerset of the outcome space) to be clear. In this way, mathematical and physical practice accommodates classic thought experiments suggesting that (metaphysically possibly) some precise propositions can lack objective (and, for that matter, subjective[12]) probability.

3 The Philosophical Question

But how are we to understand this situation philosophically?

If we just think about the probability axioms as trying to capture normative constraints on human representations of probabilities, via models and scientific theories, the choice to allow limited event spaces might seem pedestrian. For obviously, in a given practical situation, certain sets of possible worlds/outcomes might "not [be] of interest". And scientific theories attributing probabilities don't have to be maximally opinionated.

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sets. It is meaningless to ask for the probability of a subset of Ω that is not measurable. Typically, the sigma-algebra will be smaller than the power set of Ω, but large enough to include all sets of interest to us. Restricting the class of sets for which we assign probability is the key idea that will resolve the difficulties we were having with the examples of stick-breaking or infinitely many coin tosses" [15]

[11] The argument above shows we can't assign values to Vitali sets in a way that satisfies translation invariance alongside with other intuitive principles we expect area and probability of a *fair* dart landing in the region to satisfy. Note that strictly speaking problematic regions like the Vitali sets literally *can* be measured in the sense that values can be assigned in a way that satisfies the weaker conditions for being a measure (zero is assigned to the empty set of numbers/points and countable additivity), but not translation invariance.

[12] I take the above thought experiment to suggest that some propositions can lack ideal subjective probability (as there is plausibly no probability which an ideal agent knowing only the facts about the dart stipulated above should assign to that dart landing in the region corresponding to Vitali set V_1).

However things get weirder when we ask what propositions *have* (objective or subjective) probabilities – regardless of whether people chose to talk about them. For the above thought experiment suggests propositions can lack objective/subjective probabilities flat out, regardless of our interests. Thus we face an immediate question: which propositions lack objective/subjective probabilities and why?

Perhaps in the case of subjective probabilities[13], one might try to exploit reference difficulties arising from the strangeness of the propositions which seem to lack probabilities to answer this question. For Vitali uses (and arguably needs[14]) the axiom of choice to argue that some sets $V_1, V_2, \ldots$ with the relevant properties exist, rather than giving a method of concretely construction which could be used to get the expression 'V_1' to refer to a specific set. So perhaps you could say subjective probabilities idealize practices of entertaining and assigning probabilities to well behaved propositions that humans can express, and for this reason are not defined on certain sufficiently strange propositions which are deeply unsuitable to be objects of human thought or talk[15].

However, no such help applies when considering why certain propositions lack *objective probabilities* – at least if you take these at face value and regard probability facts as a primitive modal part of fundamental reality (as per SAR). For in the case of objective probability, the fact that *it would be difficult for humans to pick out or think about (or face bets whose payoffs track the satisfaction of)* certain propositions has no immediate relevance to – or ability to explain – why those propositions lack objective probabilities. And, on the other hand, it seems odd to suppose that metaphysical reality just brutely 'favors' certain propositions over others by allowing objective probabilities to attach to them.

Some alternatives to SAR, which don't regard objective probability as fundamental, might be better positioned to explain why some propositions lack probabilities along the lines above. For example, consider classic 'best system' Humean approaches to objective probability.

Humean approaches to probability maintain that all fundamental facts about

[13]In [10] Isaacs, Hájek and Hawthorne discuss a different approach to addressing the puzzle of propositions which cannot be assigned probabilities (compatible with standard assumptions about probability) in certain seemingly cogent scenarios – as it arises for subjective probability. They argue that we should assign relevant propositions *imprecise credences* and that we can do this in an attractive and principled way.

[14]c.f. the discussion of [2] in §6 below

[15]For what it's worth, I think this approach would still be quite problematic. For one thing, there's a question of whether more referentially tractable versions of the propositions in the above dart thought experiment could be found. For another thing, our ability to refer to sets of numbers (and hence of physical points) can be altered based on contingent physical events.

a possible world concern what actually happens – analyzing and/or grounding apparently modal facts about laws and probabilities in claims about the actual history of the world[16]. And Humean best systems approaches understand the truth of probability claims, in terms of these claims being consequences of whatever theory best condenses and systematizes the actual history of the world. As Lewis puts it in [12]

> The virtues of simplicity, strength and fit trade off. The best system is the system that gets the best balance of all three. ...[T]he laws are those regularities that are theorems of the best system. But now some of the laws are probabilistic. So now we can analyze chance: the chances are what the probabilistic laws of the best system say they are.[17] [18]

So a best systems theorist might explain why some propositions lack objective probabilities by saying the following.

> All there is to being an objective probability is being implied by the best theory of the pattern of actual events. And there is no obvious reason why we should expect the *best* theory (in the sense of best combining simplicity, strength and fit referenced above) to be fully opinionated. So we shouldn't expect all propositions to have probabilities.

However, the simple anti-Humean realist sees objective probabilities as fundamental aspects of physical reality in a way that blocks such an explanation[19].

[16] So (they maintain) there can't be two distinct possible worlds w and w', which agree in their total histories but differ regarding things like: their physical laws, counterfactuals or the objective probability of certain events occurring.

[17] Thus the objective physical probability at time t of some event occurring, is (something like) the probability assigned to this event by combining facts about the actual state of the world up to t with the statement of probabilistic laws which best systematizes the total actual history of the world (in the sense above).

[18] Such a view might compliment proposals in the literature on Borel's paradox, that events only have conditional probability relative to a Fregean mode of presentation[17]/ a choice of a σ algebra partitioning up all the possible outcomes into distinct events[6].

[19] One might compare this intuition that facts about human graspability/possible reference are the wrong kind of things for fundamental laws of physics or metaphysics assigning probabilities to care about to common resistance to physical theories on which 'observation' collapses the wave function.

4 Intrinsic Ineligibility of Propositions

How then should the advocates of SAR answer the question: which propositions lack probabilities and why?

At first glance, it might seem that the references to "non-measurability" in the textbooks quoted above suggest an an easy response along the following lines. Some propositions (including the Vitali ones mentioned above), are somehow intrinsically 'non-measurable' (or not Lesbague measurable) in a way that makes them unable to have objective probabilities at any metaphysically possible world.

However I think a number of problems arise when we try to seriously flesh this idea out. I take the form of the intended proposal to be something like the following.

1. Certain sets of numbers (e.g. Vitali sets) are 'not measurable' in the specific sense of having no *Lebesgue measure*.

   - As noted above, the Lebesgue measure on the reals is the unique translation invariant measure on the real numbers assigning the unit interval measure 1 and assigning some measure to intervals.[1].
   - Note Vitali sets are not literally non-measurable. There are other ways of assigning measures to sets of numbers[20] which do not satisfy translation invariance but do assign values to the Vitali sets[21] – as will be discussed below.

2. The fact that these *sets of numbers* are non (Lebesgue)-measurable somehow explains (or parallels the explanation for) why 'corresponding' *regions of physical space* (e.g. those we would pick out by reference to them using Cartesian coordinates associated with physical space) lack a well defined length/area/volume.

   For example, an absolutist about distance might think of relevant grounding facts as working like this

[20] That is, there are functions which assigning numbers to these sets in a way that assigns 0 to the empty set and satisfies countable additivity as per standard definitions of measure.

[21] I take it there is no obvious connection (in general) between translation invariance and probability (in general), thought there is one between translation invariance and our concepts of physical length/area/volume etc. Translation invariance was relevant to assigning probability in the specific thought experiment imagined, because we stipulated that the dart was fair.

- Each metaphysically possible world gets to specify what points it contains and how these are related by a unique physically favored distance function
- (At all possible worlds) this determines a correct assignment of length/area/volume to the 'easy case' of sets of points which constitute open balls with respect to the true distance function
- (At all possible worlds) a set of physical points has a length/area/volume iff it can be assigned one in a way that generalizes the assignment of these properties to sets of points that constitute open balls (relative to the physically natural distance function) as per the definition of Lebesgue measure. Lebesgue measure provides a conceptual analysis of what it means (where it is meaningful) to say an arbitrary physical region has a length/area/volume, taking facts about the true physically natural distance function as given.

3. The fact that these regions of physical space lack a well defined length/area/volume somehow makes corresponding (!) propositions intrinsically unsuited to have objective probabilities (so they will lack probabilities at all possible worlds). In particular, the fact that some region R lacks a well defined area will turn out to ensure the propositions involving that region like 'the next dart thrown will land in region R' [22] lack an objective probability (at all possible worlds)

4. All propositions (qua sets of possible worlds) have objective probabilities, except for those which 'can't' have objective probabilities due to something like the nonmeasurability considerations just referenced.

There are two problems I want to highlight.

First, how are we to spell out the notion of correspondence evoked in step 3? When does a proposition (qua set of possible worlds) count as being 'about' a region that lacks length/area/volume, in a sense which prevents that proposition from having an objective probability?

For example, in the case above we specified the relevant class of problematic propositions by considering sentences referring to non Lebesgue measurable spatial regions (corresponding to the Vitali sets). But in general, sentences mentioning many different things can be used to express the same proposition qua

[22]Perhaps we might imagine these regions characterized in terms of Cartesian coordinates around the center of a possible world, plus the physically natural distance function.

set of possible worlds. So perhaps one could try to say that a proposition is non-measurable (in a way that prevents it from having a probability) if you would need to use reference to a non-Lebesgue measurable set as a parameter to express it. But what do we mean by 'need' here (e.g., what other linguistic resources are we assuming in the background)? Philosophical work seems needed to spell this out in a principled way that avoids trivialization/counterexamples.

Second, even if we can specify the relevant notion of connection/aboutness, why should the fact that a proposition is about a set that lacks a *Lebesgue measure* (which builds in translation invariance) or spatial region that lacks length/volume/area be enough to prevent it from having an objective probability at all possible worlds — even ones where (unlike in our thought experiment) no assumptions about the probability of something happening in a region being proportionate to the area of that region, or fair in a way that motivates translation invariance are made[23]?

Furthermore, this strategy seems to require bullet biting in the form of denying that the Vitali dart propositions can have objective probabilities relative to other metaphysically possible worlds with unfair darts.

It seems fully conceivable and metaphysically possible for there to be a world much like ours/the imagined world of the thought experiment containing a dart which (is not equally likely to land anywhere on the dartboard and) *does* have a definite objective probability (say 1/2) of landing in the spatial region corresponding the to the Vitali set V_1.

Perhaps the metaphysical possibility of a dart which behaves like this (relative to the physically natural way of assigning distances) can be supported by intuitions connecting metaphysical possibility with recombination (c.f. Hume's treatise[9] and Lewis[13]). And it is possible to tell a mathematically cogent story – obeying all the probability axioms – about what the objective probabilities of this dart landing in many other regions might be[24].

[23]Note, there is no mathematical obstacle to assigning probabilities to the Vitali regions while retaining all of the ordinary probability assignments to Lebesgue-measurable regions and satisfying the axioms of probability. For Bierlein's measure-extension theorem implies that, given a probability measure (μ) and any countable family of pairwise disjoint sets $(A_1, A_2, \ldots)$, there is a countably additive probability measure extending (μ) whose event space contains every (A_i). See Bierlein [5] and Lembcke [11].

[24]For example, couldn't there be a centered metaphysically possible worlds where the dart closest to the center has: probability 1/2 of landing in the region corresponding to V_1, probability 1/4 of landing in the region corresponding to V_2, probability 1/8th of landing in the region corresponding to V_3 and generally probability $\frac{1}{2^n}$) to the region corresponding to V_n? One could further assign probabilities to a σ algebra of events which includes (all and only) the empty set of possible worlds and the sets of possible worlds corresponding to arbitrary unions of these countably many basic/atomic events in the obvious way, while satisfying the

Accordingly just saying that some propositions intrinsically/a priori lack probabilities because they are 'not (Lebesgue) measurable' does not suffice to suffice to answer the philosophical question above.

5 World-relative probability gaps

Instead, I think it is more attractive to say facts about which propositions have objective probability are world relative. That is, we might think of laws of nature at each metaphysically possible world as directly determining which propositions have objective probabilities at this world (not just what those probabilities are).

This would let us honor the Humean recombination intuitions above by saying the very same proposition (claiming a dart will landing in a dartboard region corresponding to a Vitali set) would

- lack an objective probability in the fair dart throw situation imagined in our initial dart thought experiment, but
- have and objective probability in other conceivable and metaphysically possible worlds where the physically natural assignments of distance and area are exactly analogous but the dart throw is not fair (e.g., it is certain to land on one spot, or has probability 1/2 of landing in a particular Vitali region)[25]

In this case, it will be a metaphysically contingent (and presumably empirical) question whether the claim that some dart will land in a certain spatial region has an objective probability. I think this approach has significant appeal. However we should note that adopting it amounts to a substantive philosophical move.

Adopting it also raises philosophical questions about how to think about the space of metaphysically possible worlds. What constrains how the laws of nature at different metaphysically possible worlds can select an event space? For example:

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standard probability axioms.

[25]The problem in §3 doesn't arise directly from attaching *some* probabilities to the dart landing in relevant convoluted Vitali regions of the dart board. Rather it comes from intuitions that *in the particular kind of dart throwing scenario imagined above*, this event could only have an objective probability if it had one satisfying the following combination of constraints (1) countable additivity, (2) translation invariance, and (3) assigning a non-zero probability to the dart landing somewhere on the dartboard in question. However constraints (2) and (3) need longer apply in possible worlds where the physical setup for the dart throw or the laws of nature were suitably different.

- Could there be two distinct metaphysically possible worlds which agree on their actual history, and the probability of all events that have objective probabilities in both worlds, but disagree on whether certain propositions have objective probabilities?

- Does the set of propositions which have objective probabilities at a given possible world need to be 'maximal' (among metaphysically possible worlds)? Or could there be worlds w_1 and w_2 such that all propositions with objective probabilities in w_1 are a proper subset of those with objective probabilities in w_2?

- There is (for me at least) intuitive appeal to a Humean recombination idea that there should be metaphysically possible worlds corresponding to all logically possible ways of selecting a set of some propositions (qua sets of centered possible worlds) to have objective probabilities that are compatible with a small range of a priori constraints.

  But what are the relevant constraints? One obvious idea would be to say that the only constraint is that the propositions with objective probabilities at a world must always form a σ algebra (because the Kolmogorov axioms for probability hold with metaphysical necessity).

  Once can begin to pump intuitions for this approach by saying things like the following. It is absurd to imagine a possible world relative to which there is (say) a 50% objective probability of a coin landing heads up but an undefined probability of it not landing heads up. And once a creator or the laws of nature at some possible world has succeeded in assigning a 20% probability to P and a 30% probability to a metaphysically incompatible proposition Q, there is no further gap in reality it could leave so as to make the objective probability $P \vee Q$ undefined. But perhaps this could be disputed. And perhaps it is less obvious that all countable unions of propositions with probabilities must have a probability.

  Also, could there be other relevant constraints?

So overall I think simple anti-Humean realists about objective physical probabilities have (at least) further philosophical work to do, if they want to explain which propositions lack objective probabilities by implementing the strategy sketched above.

6 Resisting the Argument that Some Propositions Lack Probabilities

Let me end this paper by briefly explaining why I'm not optimistic about attempts to cut the Gordian knot by rejecting some step in the the classic argument in §2.1 that some propositions lack objective probabilities. In this section I'll survey what I take to be the best options for doing this, and explain why I don't think they are promising.

First, you could deny that our description of a scenario where a dart is 'equally likely to land anywhere' (and hence, e.g., equally likely to land in any two regions of equal area) on a rectangular dartboard is coherent or expresses something genuinely metaphysically possible. But this seems clearly unintuitive[26].

Second, you could reject the axiom of choice or some other premises used in Vitali's purely mathematical argument for the existence of sets of numbers with the properties claimed in §2.1. As noted above, the Vitali sets V_i are not Lebesgue measurable[27]. And Solovey showed that (given the assumption that there is inaccessible cardinal), there is a model satisfying ZF and Dependent Choice (a weakened form of the axiom of choice) where every set of real numbers is Lebesgue measurable[2]. This suggests one can consistently (in a sense) block the argument above by rejecting relevant appeals to the axiom of choice[28]. However, rejecting one of the standard ZFC axioms of set theory is clearly a boldly revisionary move.

Third – and perhaps more initially attractively– you could say the above description of the dart board is cogent (and corresponds to something genuinely metaphysically possible) but reject the idea that, since each Vitali set is got by uniformly shifting V_1 over by some amount, the dart must have equal probability of landing in the physical region corresponding to each Vitali set V_i. As noted above, the Vitali sets are not (Lebesgue) measurable and you might think the

[26]Perhaps one could motivate the view that there is something deeply problematic about our intuitive idea of a dart being 'equally likely to land anywhere' by referencing the Borel paradox which concerns conditional probabilities but also appeals to translation invariance intuitions (as e.g. discussed in Esawaran's [6]).

See the appendix for a discussion of Hoek's recent arguments that we should be willing to bite this bullet in [8]

[27]The argument that they can't have a probability directly parallels the argument that they lack a Lebesgue measure.

[28]Also there are independent reasons people have doubted the axiom of choice, like the Banach Tarski Paradox.

above intuitions about uniform shifting over implicitly invoke a concept of 'area' that only applies to Lebesgue measurable regions. So you could, in principle, reconcile the metaphysical possibility of our dart scenario with the expectation that all sets of possible outcomes/propositions have objective probabilities by saying the dart has

- equal probability of landing in regions of the dartboard with *well defined and identical areas*
- different probability of landing in the spatial regions corresponding to different Vitali sets V_i (which lack well defined areas) – with the sum of these countably infinitely many probabilities choosen to satisfy the assumptions of the problem (and the probability axioms).

However, difficulties arises when fleshing out this approach with a principled position regarding which Vitali-set regions the dart is more likely to land in and why. For example, any (putatively) necessary and a priori view about which of the regions corresponding to different Vitali sets a fair dart would be more likely to land in seems arbitrary and hard to motivate[29]. Thus this move threatens to replace one challenge (why do some propositions have probabilities while others don't?) with another (why is the dart more likely to land in some regions than others?).

Fourth, you could reject countable additivity – or some other of the standard axioms of probability used to get paradox from the assumption that the dart has a well defined and equal probability of landing in a physical region corresponding to each Vitali set in §2.1. Some independent reasons for concern about countable additivity have been proposed[30], but I take the motivations for sticking with such an intuitive and well entrenched principle to be clear.

[29]It might be better to say the above description of the dart's behavior is cogent but incomplete: realizable in different ways corresponding to different choices for how to assign different objective probability to the dart landing in regions of space corresponding to the different Vitali sets. Perhaps one can motivate this view by comparing our dart scenario to Thomson's lamp[14]. One might say the question of what probability a dart with the properties described in §2.1 above has of landing in each of the regions V_i is analogous to the question of what position a light switch will end in if it starts in the "on" position and gets flicked on and off infinitely many changes within a finite interval. In both cases a description seems like it must settle the answer to a follow up question – but this description turns out to be satisfiable in different ways that yield different answers to the follow up question. However, accepting that darts which are both 'equally likely to land anywhere in the rectangle' in the intuitive sense initially evoked can differ in their probability that the dart lands in a certain region V_i still feels a bit counterintuitive.

[30]See the summary of such concerns at the beginning of [7].

So, overall, I don't see any easy way for the simple antihumean realist about objective probabilities to evade the philosophical challenge raised above by rejecting classic arguments that some propositions lack objective probabilities.

7 Conclusion

In this paper I've tried to highlight a tension between simple anti-Humean realism (SAR) about objective probability and standard arguments that some propositions lack definite probabilities. I've suggested that this tension provides an internal challenge for simple realists about objective probabilities.

Opponents of SAR might use this challenge to motivate switching to a humean best systems approach, because these seem to have more tools for explaining why some propositions lack objective probabilities.

However defenders of SAR could see this challenge as merely showing the need for significant (and perhaps surprising) additions to their view of the nature of objective probability. I have tried to highlight some problems and significant choice points for what I take to be some of the most obvious ways of explaining why some propositions lack probabilities in a way compatible with SAR. But I don't mean to advocate general pessimism. Rather, I hope to have shown how standard arguments that some propositions lack probabilities raise interesting philosophical questions and choice points, when applied to objective physical probability.

A Hoek's Defense of All Propositions Having Objective Probability

Daniel Hoek's recent paper [8] provides an interesting example of adopting the first resistance strategy mentioned in §6. Specifically he advocates maintaining that all propositions have objective probabilities, by denying that scenarios like the fair dartboard situation imagined in §2.1 above are even 'in principle' possible. He writes, "What Vitali did show is that, given the axiom of choice, countably additive, total chance functions on a continuum lack a certain kind of symmetry: more specifically, they fail to be translation invariant, …Vitali's theorem shows that totally fair spinners are not possible. This is an interesting and surprising result … [I]t was clear from the outset that a totally fair spinner

is a physically unrealistic idealization. However, it is striking that this ideal cannot be achieved even in principle."

To further motivate biting this bullet, Hoek makes a few arguments that aspects of current scientific practice somehow requires all propositions to have probabilities. I will review, and raise questions about, a few of these. First, Hoek argues that allowing propositions to lack a probability would undermine our ability to reason inductively about probabilistic events. He illustrates this using a Mystery Lamp thought experiment where a lamp has been observed to blink 81 times in 100 trials, and we are asked whether one can rationally use abduction to assign high subjective probability to the proposition that the lamp has a 80% objective chance of blinking when appropriately stimulated. He writes as follows

> "if the mechanism in the lamp were not totally chancy [i.e. if some sets of possible outcomes involving the lamp lack objective probabilities], that means we have to contend with an important additional hypothesis: for in that case, it may be that the blinking event has no chance at all. The problem is that if this is so, we have no idea what to expect. …Does [the behavior we have observed] confirm or disconfirm the hypothesis that this event has no particular chance at all? We cannot say for if the hypothesis that the event of blinking has no chance fits the data equally well as the hypothesis that the chance of blinking is around 80%, then that substantially undermines the strong justification we thought we had for our inductive predictions about the future behavior of the lamp."

However, (as stated) I'm not sure this argument shows commitment to anything worse than garden variety underdetermination of theory by evidence. If merely accepting the 'in principle' possibility of an alternative hypothesis equally compatible with all our data blocked justified belief, then most of our beliefs about the external world would be threatened. Why can't one regard our actual scientific practices of responding to observed frequencies (and other theoretical beliefs) by assigning some but not all propositions probabilities as a kind of legitimate inference to the best explanation (or theory choice based on a priori theory elegance)?

Second (and perhaps somewhat in tension with the first point), Hoek argues that the success of scientific theories attributing objective probabilities to some propositions gives abductive evidence for the claim that all propositions have

objective probabilities. But is this the case? Insofar as actual scientific theories attributing probabilities only assign probabilities to a limited event space (and not to particles being measured in vital regions) one might instead see them as motivating correspondingly limited theories of where events have probabilities. And even if we grant that the scientific usefulness of saying some propositions have probabilities might *ceteris paribus* motivate thinking all propositions had objective probabilities, the Vitali paradox surely constitutes a reason why the ceteris are not paribus (all things aren't equal).

Perhaps anticipating some concerns along these lines, Hoek maintains that '[o]ur ordinary probabilistic inductive practices assume that chance distributions are total' and argues that mathematicians' tendency to restrict probability measures to a specified event space (as motivated by Vitali-set dart type thought experiments) should not be taken as asserting that some propositions genuinely lack objective probabilities, but rather as a pragmatic choice:

> "[i]f one restricts the measure Ch to the Lebesgue regions of the circle [when ascribing chances to an imaginary situation involving a perfectly fair spinner], then one can at least get a function Ch that satisfies the equation $Ch(\rho L) = Ch(L)$ wherever Ch is defined [a translation invariance principle]. So by ignoring the non-Lebesgue sets, we gain an important symmetry. In mathematics, symmetries are crucial for reining in complexity: without them, you can't prove theorems, and your models become intractable. So mathematicians ignore non-Lebesgue subsets for the same reason they ignore most things: to achieve the symmetry and simplicity they need in order to do mathematics. This practice implies no positive judgment that these events lack a chance, merely the practical judgment that those chances are best ignored in order to manage the complexity of the situation.
>
> Mathematicians, then, have excellent reason to ignore non-Lebesgue spinner events. But we philosophers do not, at least not if we are interested in the nature of future contingency. For one, it is questionable whether it is even possible draw any sharp general boundary between Lebesgue and non-Lebesgue propositions about the future that corresponds to the mathematical distinction between Lebesgue and non-Lebesgue sets of real numbers. But let's not dwell on that difficulty. Even if we can draw this distinction, surely we should not

> ignore the non-Lebesgue spinner events. The least we can do is ask whether these events do or do not have a chance."

I take Hoeke's point to be that when mathematicians model fair spinners by describing probability functions which only assign probabilities to events involving measurable spatial regions (letting one assign probabilities in a way that preserves the symmetry principle of translation invariance), this reflects merely pragmatic focus on cases we can easily prove things about. I am sympathetic to the point that we should be cautious about reading much serious philosophical commitment into mathematical practice. And obviously it's true that mathematicians favor studying simple and elegant structures (with features like symmetry principles that make it easy to prove things about them) while allowing that other structures exist.

However it seems wrong to me to classify mathematical practice of considering limited event spaces in connection with the the Vitali dart propositions above (and analogous spinner claims) as an instance of merely focusing on portions of mathematical reality which are simple enough to prove beautiful and powerful theorems about.

For, as we have seen from the textbook presentations referenced in §2.1, mathematicians don't seem to merely ignore these propositions as potentially having probabilities but ones too complex to be worth proving things about – like some messy function on the real numbers to the real numbers that will never be mentioned in mathematical writing. Rather they consider such propositions and then positively argue that they *cannot* be assigned probabilities relative to certain conceivable possible worlds/situations involving fair darts/spinners being considered.

Furthermore mathematicians are perfectly willing to prove things about measures on non-Lebesgue measurable sets (that are elegant in other ways) in other contexts, where no claims about probability are being made. So it can't be that non-Lebesgue measurability blocks a function from being elegant enough to prove cool things about.

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