Quantifier Variance, Mathematicians' Freedom and the Revenge of Quinean Indispensibility?

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Outline

Section 1

Introduction

A Puzzle I

Mathematicians seem to know that mathematical objects exist. They say things like

- There are infinitely many primes
- There are mathematical objects which obey the axioms for natural numbers.

A Puzzle II

Even if we take mathematicians' ability to recognize which axioms are consistent (c.f. [2]) for granted, this knowledge can seem puzzling.

- We can't see, touch or taste mathematical objects.
- How could we know which axioms describe actual mathematical objects?

An Idea: Mathematicians' Freedom

- (Somehow) mathematicians can safely adopt *any* consistent collection of pure mathematical axioms.
- This idea is independently motivated by mathematical practice

  > *Reflecting on my experiences as a research mathematician ... [I was struck by] the freedom I felt I had to introduce a new mathematical theory whose variables ranged over any mathematical entities I wished, provided it served a legitimate mathematical purpose. [3]*

- But spelling this idea out philosophically has proved controversial.

Agenda

In this talk I will quickly review problems for the most popular existing explanations for mathematicians' freedom

- Set Theoretic Foundationalism
- Nominalism

and develop a different 'quantifier variance' based explanation.

Section 2

Prior Accounts of Mathematicians’ Freedom

Set Theoretic Foundationalism

Set Theoretic Foundationalism:

- The hierarchy of sets is very large, and nearly every coherent collection of pure mathematical posits can be truly interpreted as describing some portion of it.
- For instance, mathematicians' talk of the natural numbers might be understood as applying to $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots$ using certain sets encoding $+, *$.
- Results from mathematical logic ensure that every consistent first order theory has a set model [but posits introducing math structures like the natural numbers can't be purely first order, for Lowenheim-Skolem reasons].

Problems for Set Theoretic Foundationalism II

Set Theoretic Foundationalism requires arbitrary limits re: the size of the mathematical universe.

- For instance how tall is the hierarchy of sets?
- There must be a fact of the matter.
- But any such height would prevent mathematicians from introducing a structure (e.g. classes) of a larger size.

Nominalism and Problems for It

Nominalist views hold that there are no mathematical objects and either mathematicians claims like ‘there are infinitely many primes’ are :

- actually false
    - But this seems unintuitive
- have a different logical form than first appears.
    - Requires giving a different logical structure to ‘Evelyn is prim.’ and ‘Eleven is prime’.

Section 3

The Quantifier Variance Explanation

Holes

To motivate my QV proposal, first consider our talk of holes

- Holes are usually[4] taken to be distinct from objects like
- the air in a hole
- the material that the hole is in.

Introducing Hole Talk I

Imagine people who don't talk in terms of holes but only other things like chunks of matter, spatial regions etc.

- It seems they could start talking in terms of holes
- They wouldn't first need to check holes were part of fundamental ontology (or do other metaphysical investigation)
- And plausibly the same goes for other similar concepts

Introducing Hole Talk II

Starting to talk in terms of holes requires slightly changing the meaning of ‘there is’ as well as fixing a meaning for ‘hole’.

- For sentences that don’t involve the term ‘hole’ may change truthvalues e.g., (the Fregean paraphrase of)
  - ‘there are at least 12 things in the fridge’
  - or claims about the total size of the universe.
- This linguistic change doesn’t create a new object. Merely causes ‘there is’ to change meaning.

Quantifier Variance

Quantifier Variance (QV):

- There are a range of different meanings “there is” could have taken on, which all obey the syntactic rules for existential quantification.
- These senses need not all be mere quantifier restrictions of some fundamental maximally natural quantifier sense.

QV Explanation of Mathematicians’ Freedom

When mathematicians (or scientists or sociologists) introduce coherent hypotheses characterizing new types of objects, this choice behaves like an act of stipulative definition, which **can** both

- give meaning to newly coined predicate symbols and names and
- change the meaning of expressions like “there is”, in such a way as to ensure the truth of the relevant hypothesis.

Note: Compatible with different Metaontological Views

- Quantifier Variance explanations have traditionally been used by cynics about metaphysics.
- But this isn't a necessary component of the view.
- My story is also compatible with saying there's a single fundamental sense of existence (Siderian realism about metaphysics)
- It just requires that there are quantifier senses which introduce objects that don't exist in the fundamental sense.

Section 4

How This Helps

Benefits of QV

This QV explanation of mathematicians' freedom avoids

- Set Theoretic Foundationalism's arbitrariness problem.
    - Any logically coherent pure mathematical posit can succeed
- Nominalism's unmotivated different treatment of mathematical talk.
    - Ordinary utterances of "There is a number." and "There is a city." are both literally true, have analogous logical structures.

Parallel With Hole Knowledge

Note that adopting QV lets us give a parallel metasemantic explanation for both

- Mathematicians' knowledge of which pure mathematical structures exist
- Road builders knowledge of how deep a divot must be to count as a hole.

Access Problem for Holes

- Our beliefs about whether a shallow divot counts as a hole match up with objective facts
- No sensory experience tells us a metaphysically special way to draw the line.
- But presumably there is no access problem for holes.
- If we had used 'hole' differently both the meaning and our use would have changed to allow us to speak truly.

Section 5

Some Worries

Easy Knowledge of Yettis and Gods? I

Why don't people who believe in Yetis count as using a different quantifier sense and speaking truly? I will suggest that

- A version of this problem faces everyone already (whether they accept QV or not)
- The kinds of tools we'd use to solve the general problem plausibly also solve the QV-specific problem.

Easy Knowledge of Yettis and Gods? II

- Consider our knowledge of color boundaries, e.g., how red can pink things be?
- It's appealing to give a metasemantic explanation for this knowledge
  - If our practice of boundary drawing had been slightly different then we would have meant something slightly different by "pink" so we still spoke the truth.
- But (somehow) this abundance of concepts which words can mean + charity doesn't imply that it's impossible for our beliefs about how "pink" (or "carcinogen" or "witch") apply to be wrong.

Easy Knowledge of Yettis and Gods?* III

Tools for solving the general problem

- Charity isn't the only constraint on acceptable interpretation. We also want to
  - Make sense of observational practices, dispositions to assert and retract
  - Take people to be referring to more vs. less natural kinds (e.g., gold vs. gold-or-fools gold)
- Charitable interpreters prioritize making beliefs which our practice "treats as more analytic" come out true when tradeoffs must be made.

Section 6

Revenge of Quinean Indispensibility?

Recap

- As noted above, Quantifier Variance has traditionally been used by cynics about metaphysics.
- But (I've argued) Siderean realists about metaphysics can also give a QV explanation for mathematicians' freedom.
- Thus our explanation for mathematicians' freedom (prima facie) comes apart from our views on deep metaphysical questions.

Dilemma

This leaves a question about whether any mathematical objects exist on the Siderian fundamental quantifier sense (if there is one)? And metaontological realists might seem to face a nasty dilemma

- if we say that some mathematical objects are fundamental then we face a revived access problem.
- If we say that no mathematical objects are fundamental then we face a revenge of Quinean indispensability problem.

But I will argue that both horns of the dilemma are tolerable.

Option 1: No Fundamental Mathematical Objects

If we say that no mathematical objects are fundamental, then

- Shouldn't all facts be grounded in facts expressed using the fundamental quantifier sense?
- Notoriously nominalists have had great trouble providing paraphrases for our best scientific theories in terms of non-mathematical vocabulary.
- One might think that it will be similarly difficult to ground scientific facts in non-mathematical language.

Call this the revenge of Quinean indispensability.

Grounding Easier Than Paraphrasing

But grounding might be easier than paraphrasing in various ways, e.g.,

- paraphrasing ϕ requires replacing ϕ with a single sentence.
  - but ‘there is a cat’, could be grounded in (if there were infinitely many cats) the existence of Bess, Mrs. Wiskers etc.
- a fundamental maximally joint carving language need not be human learnable (e.g., might have infinitely many atomic predicates)

Option 2: Fundamental Mathematical Objects

- If we say that some mathematical objects are fundamental, we face an access problem as to how we know what those objects are.

Agnostic Platonism

But we can avoid this access problem by saying:

- although the fundamentalia may include some mathematical objects
- we should remain agnostic about which mathematical objects exist fundamentally.
- slogan: maybe some mathematical structures are metaphysically special, but mathematicians don't care, and don't need to care!

Hole Analogy

In the analogous case of knowledge of holes it's appealing to say that

- Construction workers can draw the line where they want and speak the truth,
- while begin agnostic about whether some specially natural sense of hole (maybe the topological sense) will be used in physics.

Section 7

Conclusion

So I think quantifier variance provides an attractive way of accounting for mathematicians freedom.

Section 8

Appendix

Constraints on Acceptable Stipulations

Saying mathematicians can stipulate literally any logically consistent sentence raises immediate worries:

- what about sentences whose truth would impose strong constraints on which non-mathematical statements express truths?
- what about pairs of internally consistent but incompatible sentence?

But we can address these by clarifying the scope and nature of the mathematicians freedom being claimed.

Problem of Restricting Non-Mathematical Vocabulary

Q: What about posits that imply size restrictions on the concrete world? Or constraints on how non-mathematical properties apply? e.g. imagine someone stipulating

- PA + there are no dogs
- PA + there are only countably many things

R: pure mathematical posits are always taken to have implicit quantifier restriction to a domain of pure mathematical objects being implicitly defined, and to avoid use non-mathematical vocabulary

Problem of Inconsistent Axiom Pairs

Q: What about pairs of internally consistent but incompatible mathematical posits?

R: understand mathematicians' freedom so that it requires that the the total collection of pure mathematical posits in play be consistent.

Is QV expressible without Paradox?

Does stating QV require saying that (paradoxically) there are some objects which we aren't currently quantifying over?

- No, we can phrase QV in terms of truth conditions for sentences without ever mentioning extra objects.
- For instance, an existential statement about holes can be given truth conditions that don't involve holes.
- See [1] for more on how to systematically non-paradoxically talk about quantifier senses more generous than our own.

Sharon Berry.
Modal structuralism simplified.
Canadian Journal of Philosophy, pages 1–23, 2017.

Sharon Berry.
Not companions in guilt.
Philosophical Studies, pages 1–24, 2017.

Julien Cole.
Towards an institutional account of the objectivity, necessity, and atemporality of mathematics.
Philosophia Mathematica, 2013.

David Lewis and Stephanie Lewis.
Holes.
Australasian Joural of Philosophy, 48(2):206–212, 1970.