PHYSICAL POSSIBILITY AND DETERMINATE NUMBER THEORY

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Abstract. It's currently fashionable to take Putnamian model theoretic worries seriously for mathematics, but not for discussions of ordinary physical objects and the sciences. In this paper, I will attack this combination of views in two ways. First, I'll (quickly) suggest there's an analogy between the challenge of understanding realist reference to physical possibility and that of understanding reference to the kind of logical/combinatorial possibility invoked when we say that second order quantifiers range over 'all possible subsets' or it would be 'logically impossible' for a property to apply to 0 and the successor of any number it applies to but not all the numbers. Second, I will argue that (under certain mild assumptions about the physical possibility of infinite stochastic physical systems) merely securing determinate reference to *physical* possibility suffices to rule out nonstandard models of our talk about number theory. So anyone who accepts realist reference to physical possibility faces pressure to also accept such reference to (at least) the standard model of the natural numbers.

1. Introduction

Putnam famously used the possibility of nonstandard models/interpretations (which make all the first order sentences of a theory true while interpreting the terms in that theory as applying to very different objects/structures) to raise a challenge for realist reference to mind-independent objects, and realist claims to have a categorical conception of the structure of the natural numbers.

It's currently fashionable to take such model theoretic worries seriously for mathematics, but not for discussions of ordinary physical objects and the sciences[1].

I will attack this combination of views in two ways. First, I'll (quickly) suggest there's an analogy between the challenge of understanding realist reference to physical possibility and that of understanding reference to the kind of logical/combinatorial possibility invoked when we say that second order quantifiers range over 'all possible subsets' or it would be 'logically impossible' for a property to apply to 0 and the successor of any number it applies to but not all the numbers.

Second, in the bulk of the paper, I will argue that (under certain mild assumptions about the physical possibility of infinite stochastic physical systems) merely securing determinate reference to *physical* possibility suffices to rule out nonstandard models of our talk about number theory. So anyone who accepts realist reference to physical possibility faces pressure to also accept such reference to (at least) the standard model of the natural numbers.

2. Putnam's Model-Theoretic Challenge

Let me begin by laying out Putnam's model-theoretic challenge, as it applies to our conception of the natural numbers.

The standard first order axioms of arithmetic (PA) plausibly articulate part of our concept of numbers (in delineating restrictions on how the symbols $\mathbb{N}, S, +, *, <$ can relate). However, these axioms can also be satisfied by non-standard models with a different structure and may change the truth-value of some arithmetic sentences. For instance, PA requires that every number besides 0 both have and be a successor, but this leaves open the

[1]cite: tim's paper, jared's paper, with quotes in footnote]

possibility of non-standard interpretations which (under $<$) look like the following (where each additional $*$ indicates a disjoint copy of the integers):

$$0, 1, 2, 3, \ldots, -2^*, -1^*, 0^*, 1^*, 2^*, 3^*, \ldots, -2^{**}, -1^{**}, 0^{**}, 1^{**}, 2^{**}, 3^{**}, \ldots$$

The resulting structure looks like a copy of the natural numbers followed by two copies of the integers. Note that, by ensuring there is no least 'infinite' number, such a structure can satisfy the requirement that every 'number' besides 0 both is and has a successor.

This alone isn't enough to create a non-standard model of PA[2]. However, if we instead consider the structure consisting of a copy of the natural numbers followed by infinitely many copies of the integers *densely ordered* (i.e., the resulting structure has the form $\mathbb{N} + (\mathbb{Z}) \cdot Q$ where $\mathbb{Z}$ is just the integers and Q is the rationals)[5], then there is a way for the relations $+, *, <$ to apply so that all so that all the Peano axioms are satisfied – including all instances of the first order induction schema[3]. Such non-standard models of PA don't satisfy the full second order induction axiom, $(\forall X)\,[(X(0) \wedge (\forall n)(X(n) \rightarrow X(S(n)))) \rightarrow (\forall m)X(m)]$. Indeed, the set X consisting of just the standard integers (i.e., the objects represented by '$1, 2, 3 \cdots$' in the example above) satisfies $(\forall n)[X(n) \rightarrow X(S(n))]$ but doesn't satisfy $(\forall n)X(n)$. So they don't satisfy second order Peano Arithmtic. But they do satisfy all the axioms in PA, i.e., those of first order Peano Arithmetic.

In view of the existence of such non-standard models, one can ask (as Putnam does) the following question. Do we really have a definite concept

[2]Note that we've only said that the basic first order Peano axioms about S applies are satisfied in the structure above, not that any of the other ones (i.e. those for $+, \cdot$ and $<$) are.

[3]That is, all sentences of the following form in the language of arithmetic $\phi(0) \wedge (\forall n)[\phi(n) \rightarrow \phi(S(n))] \rightarrow (\forall m)\phi(m)$.

of 'the structure of the natural numbers' which is not satisfied by any non-standard interpretations? What can such a concept consist of? What is it about us which (perhaps together with other kinds of facts about the world) lets us our words like "number" and "plus" take on meanings which rule out such non-standard models? For reasons I won't discuss here, Putnam takes our ability to give standard meanings to the first order logical vocabulary for granted in his challenge. I will follow him in doing so.

Accordingly, we can dramatize Putnam's challenge as follows. Imagine some all-knowing interpreter who is dedicated to interpreting our talk about the natural numbers in some unintended fashion. This mischievous interpreter has full access to ordinary determinate mathematics and uses that knowledge to construct non-standard models for our talk of the natural numbers to refer to.

Can we cite plausible constraints which our mischievous interpreter must honor which prevent him from giving an unintended interpretation? Note that classic results in mathematical logic [4] tell us that no further mathematical specification (i.e., extending PA or embedding the numbers in a larger structure) could provide such a constraint. Also note that people often invoke our causal contact with objects like rabbits and electrons as part of an answer to Putnam's more general model-theoretic challenge (which applies to scientific and everyday objects just as much as mathematical abstracta). But plausibly (even if it works generally) this appeal to causal contact isn't available for answering Putnam's challenge with respect to the natural numbers.

If we can give no satisfying answer to Putnam's challenge, then, perhaps, we must allow that our conception of the structure of the natural numbers is vague and allows for a range of acceptable precicifications (corresponding to different structures satisfying the Peano axioms, much like the range of acceptable precisifications of 'bald()' and 'heap()'). Hartry Field[2] has emphasized that taking such a position would still allow us to use classical logic when reasoning about the natural numbers (because, e.g., formulas of the form '$P \vee \neg P$' will be true on all acceptable precicifications). But admitting our conception of the natural numbers is vague involves significant bullet biting with regard to the idea that all statements of arithmetic (even ones we can't decide) have definite truth-values – as the most common way of ensuring such definite truth values is through reference (up to isomorphism) to the natural numbers.

3. Physical Possibility, Logical Possibility/Second Order Logic as Candidates for Intrinsic Eligibility

Personally, I think we should respond to (any mathematics-specific version of) Putnam's challenge by saying that we can grasp a logical notion of second order or logical possibility, in much the same way that we can grasp the notion of objective physical possibility. In both cases it is attractive to say that we that we can grasp the relevant notion by something like attempted to deference to reference magnets that may be in the neighborhood.

Contemporary philosophers have tended to dismiss Putnamian model-theoretic worries concerning our ability to talk about the natural world, while lending more credence to these arguments for indeterminacy in the case of mathematics [**?**][**?**] – partly because one can more plausibly appeal to causal contact to help rule out devirnt interpretations in the physical case. However, it doesn't seem that mere causal contact with actual objects (excellent as it may be for ruling out interpretations on which 'Barak Obama' refers to a set), can suffice to let us latch on to a unique notion of objective physical possibility[4].

[4]This seems especially true if you reject contentiously Humean approaches views on which facts about objective probability and physical laws in a world are determined by what most

So I think we probably need extra resources (e.g., facts about intrinsic eligibilty like David Lewis' reference magnets) to answer Putnam's model-theoretic challenges to our grasp of a realist notion of objective physical possibility, and strongly suspect that the same resources will also let us answer it for second order quantification and/or a notion of logical/combinatorial possibility (i.e., our aparant ability to think about really all ways it would be combinatorially possible for a predicate to apply to the natural numbers as opposed to merely all definable ways) with enough power to rule out non-standard models. Note that being a reference magnet isn't about causally changing people's behavior, but rather about being an intrinsically preferred target target for interpreting language. So the abstractness/causal inertness of these logical concepts doesn't prevent them from being reference magnets. Considering the paradigm example of things which have hitherto been suggested to be reference magnetic (the plus function rather than Kripke's quus function[**?**], the property of being gold vs. the property of looking golden) further suggests that modal and logical notions (like full second order quantification or logical possibility) are as good candidates for being reference magnets as anything else is.

But I don't claim to establish this relationship between grasping objective physical possibility and grasping full second order quantification/combinatorial possibility here[5]. There is much debate on the topic and the very idea of reference magnetism (i.e., of some concepts being more intrinsically eligible targets for interpretation than others) has raised some powerful worries[6]

elegantly surprises the pattern of actual events, e.g., coin tosses, measurements etc in that world.

[5]That is, when quantifying over all logically possible subsets.

[6]For example, there is a kind of access worry for reference magnetism facts, which is powerfully developed by Tim Button in [**?**], and positing reference magnet facts can seem to require going far beyond anything the sciences give us. However I think the ordinary practice of science involves a kind of implicit reasoning about joints in nature. For we assume that nature is uniform in the sense that it allows for elegant laws *when described in sufficiently joint carving vocabulary (e.g. green vs. grue)*, and we take certain kinds of

Instead I will attack the fashionable combination of accepting Puntamian model-theoretic arguments for mathematics but not the physical world in a more modest and direct way. I will show that merely taking us to (somehow) have a determinate grip on physical (or metaphysical possibility) provides sufficient resources to answer Putnam's challenge concerning arithmetic. Our ability to form a definite notion physical (or metaphysical) possibility provides a kind of 'failsafe' which is sufficient to rule out non-standard interpretation of our talk about the numbers on its own.

4. Contrast With Open-Endedness and Temporal Approaches

Let me begin by quickly reviewing the two closest proposals to mine in the existing literature.

4.1. The Language Expansion Approach.

In [7] [6] Parsons and McGee have offered an answer to the Putnamian challenge, centering on what McGee calls **openendedness**: the idea that we expect all instances of the first order induction axiom schema to continue holding true in any "logic preserving" extension of our language.

McGee argues (roughly[7]) as follows. McGee argues that part of our current use of number talk is to expect that the induction schema will remain true in all 'logic preserving' expansions of our language. McGee suggests that this fact helps rule out non-standard models as follows. Suppose (for

experience to justify modifying our presumptions about joint carvingness (e.g. suggesting that terms like 'alive' and 'down'). And I think there's a close relationship between this kind of joint carvingness and reference magnetism: we might articulate a reference-magnetic answer to Putnam's challenge by saying that good interpretations favor assigning people concepts which are relatively more joint carving, even if this means concluding that their ideal science would get things wrong (just as they favor assigning people's names to reference to concrete objects which have a particularly close causal relationship to tokenings of these names). So I'm fairly optimistic about providing a satisfying epistemology beliefs about reference magnetic joints in nature, which is continuous with (if not actually part of) the epistemology of scientific induction. But that is a project for a different paper.

[7]McGee's actual proposal is somewhat more complicated in ways that I claim don't effect any of the criticisms discussed here. See pgs 56-68 of [6] for the details I've elided.

contradiction) that some nonstandard model M provided an acceptable interpretation of our terms 'natural number', 'successor' etc. Then there could (in some sense) be a god who is able to point to the non-standard model and introduce a term "smee" which applied counter-inductively to this non-standard model (i.e., smee applies to 0, and $smee(n) \rightarrow smee(S(n))$, but smee doesn't apply to every 'natural number'). If we met such a god then we could (logic-preservingly) extend our language by taking the term 'smee' from their language and adding it to ours. In such a case, we would still expect the induction axiom to hold for formulas involving smee which we got from talking to this god. Therefore, interpreting us to mean a nonstandard model is unacceptable because it would fail to satisfy induction in this extended language.

This strategy faces a number of objections. First, it might well be metaphysically impossible for a god to introduce a term like smee. For instance, it's not clear how the god could refer sufficiently definitely to some proper initial segment of our non-standard model. What can the god do to secure reference in a way we cannot? Are we to imagine a metaphysically impossible scenario where they fly into the realm of abstract objects and point one by one to each of the infinitely many elements in the initial segment? Maybe we should imagine they perform some supertask with physical objects that pins down this initial segment[8]. Perhaps this objection is close to what Field and in mind when he expressed a worry like, 'why can't we just say that

[8]If the god just introduces standard explicit definitions this doesn't seem to increase the expressive power. Maybe we are supposed to imagine them introducing a truth predicate for our language and then another on and on. But there's much debate over what happens with such truth predicates. Relatedly, the usual answer to Kaplan's paradox (if there are α worlds then there are 2^{α} possible propositions so, e.g., it can't be the case that for each proposition there is a distinct possible world at which only that proposition is expressed) provides strong reason to think many 'combinatorially possible' ways a language could work are actually *not* metaphysically possible for anyone to have.

we secure definite reference by whatever we are imagining the god to do to secure her reference?' in [2][9].

I don't think McGee would be much troubled by this worry, because he seems happy to accept the metaphysical impossibility of the scenarios he envisages and instead appeals to a the idea that we are committed to the first order induction schema being true in all *logically* 'possible' extensions of our language. He writes:

> To say what individuals and classes of individuals the rules of our language permit us to name is easy: we are permitted to name anything at all. For any collection of individuals K there is a logically possible world-though perhaps not a theologically possible world-in which our practices in using English are just what they are in the actual world and in which K is the extension of the open sentence 'x is blessed by God'. So the rules of our language permit the language to contain an open sentence whose extension is K[6].

However, one might worry that our dispositions in metaphysically impossible scenarios like the above are not clearly enough understood to be invoked in this context. Additionally, one might worry that such counter-possible conditionals don't so much explain our ability to determinately refer to the natural numbers as package the intuition that we do.

[9]Field writes, "...how can adopting McGee's rich view of schemas help secure determinacy? That view of schemas merely allows me to add an instance of induction whenever I add new vocabulary. But the relevant vocabulary for McGee's argument would seem to be 'standard natural number', and we've already seen that that is no help. Of course, it's true that if I could add a predicate that by some magic has as its determinate extension the genuine natural numbers, then I will be in a position to have determinately singled out the genuine natural numbers. That's a tautology, and has nothing to do with whether I extend the induction schema to this magical predicate. But if you think that we might someday have such magic at our disposal, you might as well think we have the magic at our disposal now; and again, it won't depend on schematic induction. So the only possible relevance of schematic induction is to allow you to carry postulated future magic over to the present; and future magic is no less mysterious than present magic."

One might also object that availing ourselves of the space of all logically possible extensions of our language to explain how we have a determinate conception of the natural numbers is question begging. We wouldn't accept an answer to Putnam's worries that *just presumed* we have a determinate conception of second order quantification[10] and it's not clear that considering all logically possible linguistic extensions is materially different. If we can somehow intend that the induction schema remain true in all logically possible 'logic preserving' expansions of our language (in the above sense which includes languages corresponding to all possible ways of choosing a subset of individuals for a predicate to apply to), why can't we use the same faculty to directly intend that our second order quantifiers range over every possible subset. One might also doubt that we even have a definite conception of a logic preserving extensions of our language.

4.2. **Appealing to the Actual Structure of Time.** In [**?**] Hartry Field (rather ambivalently) proposes an alternative account, on which he argues that *if* time forms a genuine ω sequence[11] (i.e., time has infinite duration and there are only a finite number of seconds between any two times) then our belief that this is true can be used to rule out nonstandard interpretations of our number talk (given standard interpretations of our temporal and event talks). I don't think this proposal works, even if the structure of time in our universe does happen to form a genuine ω sequence in the way Field imagines. For we treat the assumption that time forms a genuine ω sequence as (at best) a contingent hypothesis, and not a conceptual truth

[10] As noted above, I think that combining appeal to some such powerful logical vocabulary with explicit acceptance of commitment to intrinsic eligibility/reference magnetism facts is ultimately the right way to go.

[11] An ω sequence refers to a collection of elements which, under some relation $<$, has the same structure as the intended model of the natural numbers, i.e., is comprised of a first element, the successor of that element and so forth. Note that the claim time forms an ω sequence (assuming it is linearly ordered) is equivalent to the claim that if we start marking off one second intervals at any point those marks form an ω sequence.

constraining what we mean by 'the natural numbers'. So it's not clear that any acceptable interpretation of our language must make the structure of the numbers come out the same as the structure of time. If I believed that the number of gumballs in the jar is 70, *this* belief presumably wouldn't commit a mischievous interpreter to interpret the concept 'natural number' in such so that this statement come out true. So why would the above conjectures about the relationship between the natural number structure and that of time have any more power to constrain acceptable interpretations of our natural number concept? One might also worry about Field's talk of isomorphism and functions in this context where we are not taking the meaning of second order quantifiers for granted[12].

5. My Proposal

5.1. **Expectations about physical possibility instead.** I will now present a different kind of answer to Puntam's model-theoretic challenge to our ability to form a definite conception of the natural number structure, which avoids all the difficulties above. Specifically, I will argue that if we are are somehow able to latch on to a notion of *physical possibility*, and certain kinds of infinite chancy structures (in a sense to be defined below) are physically possible, this suffices to rule out nonstandard models of number theory. Thus, although I think we grasp the natural number structure directly via employing certain intrinsically eligible logical vocabulary (like second order quantification or the logical possibility operator discussed in [**?**]), I will argue that grasping a notion of physical (or metaphysical possibility) provides a kind of failsafe or back door to grasping this notion.

[12]However if Field accepted something like the trick I propose below, then we could go from the fact that time is actually an omega sequence realist reference to time and expectations that no definable predicate applies counter-inductively to an ability to mean the standard models.

5.2. **Assumptions.** Let me begin by laying out the following key assumptions which my argument requires. First, we need a bunch of fairly bland assumptions which (mostly) concretely cash out the idea that we are taking definite reference to objective facts about physical necessity for granted.

I will say that an interpreter is **well behaved** iff they satisfy the following criteria:

- They select a single model as the referent of our concept 'natural number' at all physically possible worlds.
- They cannot tamper with extension of the following non-mathematical vocabulary: 'coinflip' 'heads' 'temporally after' at any of these physically/metaphysically possible worlds.
- They give the usual meaning to logical vocabulary and physical (or metaphysical) necessity operator, e.g., the existential quantifier and the physical necessity operator $\Box_p$ must contribute to truth conditions in the usual fashion. However, the interpreter is free to choose any model for the natural numbers by selecting a domain (the objects masquerading as the natural numbers), a 'natural number' for the constant 0 from that domain and functions $S, +, *$ on the 'natural numbers'[13].
- They must make all statements which we are willing to endorse as conceptually required by our grasp of the natural numbers (such as the Peano axioms) come out true. Note that this requirement extends beyond purely mathematical vocabulary by extending the induction schema to encompass any total (definitely true or false for every input) property on the 'natural numbers' describable in

[13] We will also make use of the relation $<$ but regard $x < y$ merely as an abbreviation for $(\exists z)(x + S(z) = y)$.

our current language[14]. Thus, if $Q(n)$ abbreviates "there is an n-th coinflip, and the n-th coinflip comes up heads,", then from $Q(0) \wedge (\forall n)\,[Q(n) \implies Q(S(n))]$ we can infer $(\forall n)Q(n)$

- They vindicate all sentences we treat as conceptual truths relating the numbers and the practice of counting a sequence of events in time (specifically it suffices to vindicate those truths specified in section 5.3).

I will argue that (if the substantive physical assumption below is true) any well-behaved interpreter in the sense above must interpret us as talking about a standard model of the natural numbers.

Secondly we need the substantive assumption that it is physically possible for there to be an infinite series of chancy events. I don't claim this principle is obviously true, but I think it is plausible enough to make my result interesting/worrying for those who would combine realism about physical possibility with indeterminacy about our conception of the natural numbers.

> Infinite Random Sequence (IRS): It is physically possible to have a series of independent *objectively* random events linearly ordered in time[15] with two possible outcomes ('heads' and 'tails') having a first event but no final event. Furthermore, every event in the series has a temporal successor, i.e., for any event x there is some other event y occurring after x such that no event z occurs between x and y.

[14]More restrictively, it suffices to augment the mathematical vocabulary with the terms in 5.2 and 5.2

[15]That is for any distinct events x, y in the series either x occurs before y or y occurs before x. Moreover, from the point of view of relativistic physics, the measurements are separated by time-like intervals (x is in the future lightcone of y or vice versa) so all observers agree on their order. Given these constraints it is safe to simply work relative to some fixed inertial reference frame and ignore relativistic complications for the remainder of the paper.

Informally, one can think of the events whose possibility IRS asserts as being like the ticks of an indestructible watch which never needs repair or winding. There is a first tick, each tick is followed by a unique next tick and there is no tick after which the watch breaks down.

To motivate accepting this principle, note that it is only asserting that it is physically possible to repeatedly perform (independent) textbook spin measurements on an electron[16][1] (or some other equivalent process) and that the laws of physics don't rule out time continuing infinitely into the future (though possibly having non-standard 'length')[17]. I will abstract away from the details of the measurement and simply refer to it as a 'coinflip' and the two outcomes as 'heads' and 'tails.'

5.3. **Banishing Non-standard Models.**

With these assumptions in place, we can finally turn to foiling the mischeveious puntamian interpreter. My argument will turn on the following key claim, which I think is a quasi-analyticity in the sense that its clear enough why a Putnamian interpreter needs to make it true if they need to make PA_2 'it's physically impossible for the n-th coinflip to land heads for 0 and the successor to every number, but not for all numbers'.

To write this claim up carefully in first order logic, note that our current mathematical language allows us to use natural numbers to talk about events taking place in time such as 'the 4th U.S. President' or 'the 37th successful rickrolling'. This practice of talking about the nth coinflip presumably includes accepting principles like, 'if no coinflip occurred before x,

[16]That is perform a spin measurement along the x-axis on an electron whose spin has just been measured (and thus collapsed) along the y-axis. Thanks to REDACTED for suggesting these details.

[17]We will see that, ultimately, the use of objective randomness is just a way to establish it would be physically possible for there to be a temporal ω sequence of of objects satisfying some property (having a determinate extension) in our current language.

then x is the 0th coinflip.' I take the following such principles to be conceptual truths regarding counting (temporal) sequences of events using the natural numbers[18], where $coinflip(x)$ denotes x is a coinflip, $countflip(n, x)$ denotes x is the n-th coinflip, $heads(x)$ denotes that coinflip x has the heads outcome and $before(x, y)$ denotes that the coinflip x occurs temporally prior to coinflip y.

- An object x is the 0th coinflip, i.e., $countflip(0, x)$ iff x is a coinflip and all other coinflips happen after x. $(\forall x)[countflip(1, x) \leftrightarrow coinflip(x) \wedge (\forall y)(countflip(y) \rightarrow before(x, y) \vee x = y))]$
- If x is the nth coinflip, then y is the $S(n)th$ coinflip iff y occurs after x and no other coinflip occurs between x and y. That is,

$$(\forall n, x, y)(countflip(n, x) \rightarrow$$

$$[(countflip(S(n), y) \leftrightarrow coinflip(y) \wedge before(x, y) \wedge (\forall z)\neg\,(coinflip(z) \wedge before(x, z) \wedge before(z, y))]$$

- Only coinflips can be the nth coinflip, i.e., $(\forall x)(\exists n)(countflip(n, x) \rightarrow coinflip(x))$
- No two distinct numbers correspond to the same coin flip. $(\forall n)(\forall m)[coinflip(n, x) \wedge coinflip(m, x) \rightarrow m = n]$

I take it that we accept all these principles, and take them to apply with physical (and metaphysical and logical etc) necessity. So we accept $\Box_P$(COUNTING RULES), where COUNTING RULES is the conjunction of the claims above.

For suppose the mischievous interpreter wants to take us to refer to some non-standard model of the numbers. Together with IRS, the above conceptual truths regarding counting ensure[19] that (at some physically possible

[18] C.f. [3].

[19] Note that in many physically possible situations there will be a 'number' n such that these analyticities plus the facts about how $countflip()$, $coinflip()$ and $before()$ apply insure that there is no nth coinflip for certain values of n. For example, if no coinflips take

world) our current vocabulary lets us pick out a counter-inductive[20] collection of numbers (thereby witnessing that the restrictions on our interpreter should have prevented that choice of non-standard model) .

By IRS there is a physically possible world w where infinitely many coinflips (linearly ordered by temporally before) take place and all and only the initial ω sequence of these coinflips come up heads. I claim that the above constraints on the mischievous interpreter ensure that she takes $P(n) \stackrel{\text{def}}{=} (\exists x)(countflip(x,n) \wedge heads(x))$ to hold for just those n in the standard initial segment of the nonstandard referent of the natural numbers – so that induction fails at this physically possible world for the property $P(n)$ (expressed in terms of determinate concepts in our current language as evaluated at that world).

The mischievous interpreter can't simultaneously satisfy the induction axiom for P, and the principles governing countflip above (while taking $coinflip()$ and $heads()$ $before()$ to have their intended extension) at this troublesome world w! To see why, imagine her predicament when choosing an extension for 'countflip' in w. The principles governing countflip tell us that 0 has to be assigned to the temporally first coinflip in w, 1 to the next, and so on for all the objects in the standard initial segment of the nonstandard model. Thus we have P applying to the standard initial semgent of our nonstandad model. Now to make the induction axiom come out true we'd have to find some objects n to relate by countflip to additional numbers (either to make the antecedent that P applies to objects that are closed under successor false, or to make the conclusion that it applies to all 'numbers'

place after the nth coinflip there will be no [math mode] n+1th coinflip. Even in worlds whose possibility is asserted by IRS it might be that there are only standard temporal durations, e.g., n-seconds after only makes sense for standard integers n, in which case those worlds wouldn't have any n-th coinflip where n is non-standard.

[20]These conceptual truths do not necessarily uniquely determine which elements n in the nonstandard model will be interpreted to satisfy '$(\exists x)coinflip(x,n) \wedge heads(x)$ but they insure that all such interpretations will be counter-inductive.

true). And our axioms for nth (which she must also make come out true at all possible worlds) say that countflip relates each number to a different coinflip. So now the fact that we have already 'used up' all the ω sequence of successive coinflips that actually turned out heads on the standard initial segment of our model prevents the mischievous interpreter from making P come out true of the extra nonstandard 'numbers' in their model.

So (to summarize) if IRS is true than an instance of the induction schema fails to hold with physical necessity, if natural number has a non-standard interpretation (while coinflip(), heads(), before() have their usual interpretation).

Note that exactly the same argument would work if we replaced appeal to a definite notion physical possibility $\Box_p$ with appeal to a definite notion of metaphysical possibility $\Box_m$ – with the cheering improvement that the version of IRS which merely asserts metaphysical rather than physical possibility is clearly true. It is clearly metaphysically possible for there to be an infinite stochastic sequence.

6. Conclusion

In this paper I have argued that we can appeal to expected relationships between mathematical facts and physical or metaphysical possibility to rule out non-standard models of our number theoretic talk. I have also reviewed some worries for previous 'indefinite extensibility' based accounts of our ability to grasp a fully definite concept of the intended structure of the numbers, and noted that this approach avoids them.

Let me close on a note of humility with two caveats. First, I admit that any philosopher of mathematics who doesn't think there are determinate right answers to all questions in number theory will be inclined to doubt our determinate grip on physical and metaphysical possibility which my response assumes. Rather, my aim is merely to argue that realism about physical and/or metaphysical possibility creates a lot more pressure to be truth-value realist about mathematics than many metaphysicians realize. Second, it would be strange if our possession of a definite conception of the natural numbers depended on our beliefs about physical (or metaphysical) possibility. Thus, I think that another - rather different- style of answer to Putnam's challenge must also be possible.

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