ON POTENTIALIST SET THEORY AND JUSTIFYING REPLACEMENT

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Abstract. The ZFC axioms provide a very widely accepted foundation for set theory, and in turn for all of mathematics. Thus one might hope that, once we adopt a good philosophy of set theory, all the ZFC axioms will seem clearly true, or at least justifiable via principles that are obvious (or obvious relative to the iterative hierarchy conception of sets). However there is a long standing controversy over how to justify one part of the ZFC axioms, namely the axiom schema of Replacement. In this paper we offer a new justification for the axiom of Replacement based on a potentialist approach to set theory.

1. Introduction

Mathematical proofs is often cited as a touchstone of clarity and convincingness. The ZFC axioms provide a very widely accepted foundation for set theory, and in turn for all of mathematics. Thus one might hope that, once we adopt a good philosophy of set theory, all the ZFC axioms will seem clearly true, or at least justifiable via principles that are obvious (or obvious relative to the iterative hierarchy conception of sets). However there is a long standing controversy over how to justify one part of the ZFC axioms, namely the axiom schema of Replacement.

In this paper we will argue that philosophers who adopt a potentialist approach to set theory can provide a satisfying justification for the Replacement schema. Potentialism is a popular approach to set theory (developed by Putnam[13], Parsons[9] etc.), which lets us honor certain extendability intuitions while avoiding attributing arbitrariness to the mathematical universe, by understanding set theorists' apparent quantification over a complete hierarchy of sets as instead talking about how it would be possible for various initial segments of the cumulative hierarchy of sets to be extended.

In section 2 we will review the Axiom Schema of Replacement, and known problems for existing (actualist) attempts to justify it. In section 3, we will review the general nature and motivations for potentalist set theory, and discuss problems for existing potentialist attempts to justify the axiom of replacement. In section 4, we will specify the particular formulation of potentialism used in this paper. In section 5, we will articulate an intrinsically appealing simultaneous realizability principle (which we call 'the Hedgehog principle')[1]. Finally, in 6, we will show how potentialists can use this principle to justify the axiom schema of replacement.

2. Justifying Replacement: a problem for everyone

On a straightforward actualist approach to set theory, there are abstract objects called the sets, much as there are abstract objects called the natural numbers. We can ask, what sets exist? What kind of structure do they have under the relation of membership? Naively one might want to say that, for any property $\phi(x)$, there is a set whose elements are exactly those objects that satisfy ϕ. But, as Bertrand Russell famously showed, this leads to paradox via the conclusion that there must be a set whose elements are exactly those sets which aren't members of themselves.

The (widely embraced) iterative hierarchy conception of the sets solves this problem by considering the sets as being constructed in layers, with sets at a given layer in the hierarchy only containing elements from earlier layers. Following Boolos[1] one can spell out this idea out by saying that the hierarchy of sets consists of a two sorted structure consisting of:

- a well-ordered series of stages[2], and
- a collection of sets formed at these stages, such that a set is formed at a stage iff its members are all formed at earlier stages.

One can think of this as a partial specification for the hierarchy of sets: specifying a precise width for this structure (as all sets possible to form at a given stage are formed) but not a unique height (since there are many different logically possible well orderings which do not have a last element e.g. ω, $\omega + \omega$ etc.).

But, as Boolos points out in the same paper, the axiom of replacement doesn't obviously follow from the iterative hierarchy conception of the sets. So (even if we take for granted that there are objects satisfying the iterative hierarchy conception of sets), if we want to justify use of the ZFC axioms, a question remains about how to justify the axiom of replacement.

Informally, the axiom schema of replacement tells us that the image of any set under a definable function is also a set. More formally, let ϕ be any

[1]The Hedgehog principle cited here comes from an unpublished monograph by one of the authors in which potentialist versions of all of the axioms of ZFC are justified from intuitive axioms of logical possibility.

[2]Boolos's conception requires that the stages have no last element but, as we will concern ourselves with initial segments of the hierarchy of sets (and every well-order can be extended to a well-order with no last element) we drop this requirement.

formula in the language of first-order set theory whose free variables are among $x, y, I, w_1, \dots, w_n$. Then the instance of axiom schema of replacement for this formula ϕ says the following:

$$\begin{aligned}&\forall w_1 \forall w_2 \dots \forall w_n \forall a \\ \text{(Repl)} \qquad &\quad \forall x(x \in a \to (\exists ! y)\phi(x, y, w_1, ...w_n)) \\ &\quad \implies \exists b \forall x\big(x \in b \to \exists y(y \in b \wedge \phi(x, y, w_1, ...w_n))\big)\big]\end{aligned}$$

In other words, if we think of the formula ϕ as specifying a definable function f with a as its domain, and $f(x)$ is a set for every x in that domain, then there is some set b which contains all these sets $f(x)$ in the range of f. One can think of the Schema of Replacement as imposing a kind of closure condition on the height of the hierarchy of sets.

While we will see that justifying the replacement axiom from an intuitively attractive conception of the sets has proved difficult, the practical utility of this schema is evident. However, the fact that this schema doesn't follow from the iterative hierarchy conception raises a worry about how to justify it, and (indeed) whether mathematicians are justified in using it at all.

There has been much interest and sympathy with this worry in the subsequent literature. For example, [12] Putnam writes, "Quite frankly, I see no intuitive basis at all for . . . the axiom of replacement. Better put, I do not see that a notion of set on which that axiom is clearly true has ever been explained." And in a discussion of the history of set theory Michael Potter remarks that, "it is striking, given how powerful an extension of the theory replacement represents, how thin the justifications for its introduction were" [10] [3], and then reports that, "In the case of replacement there is, it is true, no widespread concern that it might be, like Basic Law V, inconsistent, but it is not at all uncommon to find expressed, if not by mathematicians themselves then by mathematically trained philosophers, the view that, insofar as it can be regarded as an axiom of infinity, it does indeed, as von Neumann ... said, 'go a bit too far"'

To our knowledge three main (actualist) strategies for justifying replacement are currently popular.

[3]He supports this assessment by quoting "Skolem (1922) gives as his reason that 'Zermelo's axiom system is not sufficient to provide a complete foundation for the usual theory of sets', because the set $\{\omega, P(\omega), P(P(\omega)), ...\}$ cannot be proved to exist in that system; yet this is a good argument only if we have independent reason to think that this set does exist according to 'the usual theory', and Skolem gives no such reason. Von Neumann's (1925) justification for accepting replacement is only that in view of the confusion surrounding the notion 'not too big' as it is ordinarily used, on the one hand, and the extraordinary power of this axiom on the other, I believe that I was not too crassly arbitrary in introducing it, especially since it enlarges rather than restricts the domain of set theory and nevertheless can hardly become a source of antinomies. (In van Heijenoort 1967, p. 402)".

First, people have tried to justify using the axiom of replacement 'extrinsically' in much the way that one would justify a scientific hypothesis, by appeal to its fruitful consequences, arguing it helps prove many things we independently have reason to believe and hasn't yet been used to derive contradiction or consequences we think are wrong.

Second, people have tried to justify Replacement by noting it follows from a set theoretic reflection principle, which says that for any finite list of formulas there is an initial segment of the hierarchy of sets V_α which behaves like the whole hierarchy of sets with regard to these formulas, i.e., for any objects $a_1...a_n$ in V_α the formula $\phi(a_1...a_n)$ is true[4] (i.e., true when its quantifiers are interpreted to range over the total hierarchy of sets iff it is true within V_α).

Third people have tried to justify Replacement by appeal to the idea, motivated by Russell's paradox, that size is the natural choice for the limitation on what pluralities count as sets and it should be the *only* such limitation [10]. However, this justification seems to presume rather than support the consistency of replacement with the other axioms since, the inconsistency of replacement would presumably witness that some pluralities which would be paradoxical as sets have the same size as sets.

However, while these strategies clearly succeed in providing *some* justification for using the axiom of replacement[5], we think it will be agreed that none of them account for the kind of prima facie obviousness we expect from mathematical axioms. To assure ourselves that our mathematical axioms are consistent (as well as for aesthetic reasons) we usually want these axioms to clearly follow from some simple intuitive conception which strikes us as obviously free of contradiction. For instance, we think of number theory as describing the sequence built by starting at 0 and continuing to add successors forever. We think of the reals as describing an infinitely divisible line extending infinitely in both directions without gaps[6]. The iterative hierarchy conception offers such a conception for set theory without replacement but we are left with only the relatively weak arguments above to convince us that replacement is consistent and appropriate to accept as an axiom.

We suggest that by moving to a potentialist approach to set theory its possible to provide the kind of simple intuitive conception from which replacement follows that actualism has so far been unable to provide.

[4]This entails replacement since if $(\forall x \in i)\exists! z) phi(x, i)$ is true in V then it must be true in some V_α and hence there is a set y containing the image of i under ϕ in $V_{\alpha+1}$.

[5]Specifically our failure to yet derive contradiction from replacement surely offers partial justification.

[6]One can think of a Dedekind cut which doesn't correspond to a real number as a kind of gap, i.e., a vertical line passing through the x-axis that somehow misses every real number.

3. Potentialist set theory

So much for general (actualist) problems with justifying the axiom schema of replacement. Now let's turn to the alternative potentialist approach to set theory which this paper will argue has the potential to provide a more satisfactory justification for our use of replacement.

In this section we will introduce and motivate the potentialist understanding of set theory, reviewing how it lets us address classic worries about arbitrariness and indefinite extendability which beset actualist set theory. Then we will note that, just like actualists, potentalists face a serious problem about justifying the axiom of replacement, although this problem takes a different form (since the potentialist takes an instance of the replacement schema to express a vary different claim than the actualist does).

3.1. **Potentialism as an antidote to arbitrariness.** The most popular and important motivation for potentalist set theory comes from a kind of indefinite extendability intuition, together with the desire to avoid arbitrariness.

Wright and Shapiro[16] articulate this indefinite extendability intuition as follows. For any (ordinal) height the set theoretic hierarchy might have, the very same principles which made us think there are any sets in the first place motivate thinking that the sets continue past that point. Thus, positing that the actualist hierarchy of sets just happens to stop at some height (completely underdetermined by the iterative hierarchy conception of sets or other aspect of our conception of our sets) seems like positing unattractive arbitrariness in the universe.

Intuitively, for any logically possible well ordering, it is logically possible to have a strictly larger one which adds a single object on top (and also to extend any collection of objects satisfying the iterative hierarchy conception above by adding another layer). Thus it can seem deeply unparsimonious and arbitrary to posit a metaphysically special (and perhaps unknowable) size at which the hierarchy of sets in Plato's heaven stops, a special size beyond which collections of objects are unable to form sets. Why add this kind of extra brute joint in nature to our total picture of the world, if it can be avoided?

3.2. **The core idea of potentialism.** Adopting a potentialist approach to set theory (along lines suggested by Putnam [11], Hellman[4], Parsons[9] and Linnebo[7]) lets us above problem about commitment to an arbitrary metaphysically special size. On this approach, mathematicians' claims which appear to quantify over sets should really be[7] understood as claims about how it is (in some sense) *possible* to extend initial segments of the hierarchy

[7]Strictly speaking, we take it, Putnam would say these claims *can* be so understood.

of sets, i.e., collections of objects which satisfy our intuitive conception of the width of the hierarchy of sets. Hellman, who presents (arguably) the most developed version of potentialism hitherto, understands the relevant notion of possibility in terms of a notion of logical possibility (and we will follow him in so doing)[8].

The potentialist takes set theorists' singly-quantified existence claims, like $(\exists x)(x = x)$, to really be saying that that it would be possible for a collection of objects V_0 to satisfy the width requirements in the iterative hierarchy conception of sets while containing a suitable object x (in this case, an x such that $x = x$). The potentialist takes set theorists' universal statements with a single quantifier, like $(\forall x)(x = x)$, to really say that it is necessary that any object x in any possible collection of objects forming an initial segment would have the relevant property.

The potentialist handles nested quantification by making claims about how initial segments V_α of the hierarchy of sets could be extended. For example, they will translate $(\forall x)(\exists y)(x \in y)$ as follows: if V_1 is an initial segment and includes a set x, then it is logically possible for an initial segment, V_2 to extend[9] V_1 containing a set y such that $x \in_2 y$ (where $\in_2$ is the sense of $\in$ relevant to V_2). Writing this out formally in terms of Hellman's notion of logical possibility gives us the following sentence (implicitly restricting V_1 and V_2 to range over initial segments and using $V_1(x)$ to indicate that x is a set in V_1)):

$$\Box(\forall V_1)(\forall x)[V_1(x) \implies \Diamond(\exists V_2)(\exists y)(V_2(y) \wedge V_2 \geq V_1, \wedge x \in_2 y)]$$

Note that by adopting a potentialist understanding of set theory, we avoid commitment to arbitrary limits on the intended height of the hierarchy of sets[10]. We also avoid the assumption that there is (or could be) any single structure which contains ordinals witnessing all possible well-orderings,

[8]However a number of other approaches are possible. See, for example, the closely related accounts given by Linnebo [6] and Parsons [8]

[9]By this we mean that every element of V_1 is an element of V_2 and the second order relation quantifier Hellman uses to give $\in$ its meaning on V_2 agrees with $\in$ on V_1 (and, indeed, any element of V_2 which is $\in_2$ an element of V_1 is also in V_1).

[10]More specifically, adopting potentialism lets us accept the 'indefinite extendability' intuition above that for any collection of objects that simultaneously exist (including any structure of mathematical objects), it would be logically coherent for there to be a strictly larger structure which adds new objects forming a layer of sets/classes to these original objects under some relation '$\in$', *without saying* that there there is some point at which the hierarchy of sets just stops, even though it would be logically coherent for it to go up higher. It lets us say that the full range of modal facts about extendability are relevant to set theory.
In contrast, Actualists seem forced to arbitrarily designate a certain logically possible V_α structure as the one true structure of the sets in Plato's heaven, while denying a similar place to equally coherent extensions of V_α satisfying all the same axioms.

though every possible well-ordering is realized in some possible initial segment of the sets.

3.3. **Potentialism and Justifying Replacement.** Unfortunately however, just adopting potentailism doesn't obviously banish the worries about justifying use of the Replacement Schema above. If we adopt potentialism (as we advocate), set thoeretic axioms turn out to express claims about logically possible extendability.

It is relatively straightforward to justify the potentialist translation of many ZFC axioms, using from intuitive principles about logical possibility (though we will not present such justifications here). However, the translation of the axiom of replacement can't be so easily justified. Hellman highlights this issue, and suggests that we adopt all potentialist translations of instances of the replacements schema as brute axioms, justified by their usefulness in reasoning about other things. Thus, it might seem like the potentailist is forced to give the kind of wimpy 'external' justification for replacement schema which actualists are.

Happily however, it turns out that we can justify the potentialist version of the axiom of schema of replacement on the basis of principles about logical possibility that are intuitively obvious and constitute the kind of unified conceptual whole (like the numbers or the cumulative hierarchy without replacement) which can provide confidence that a theory is coherent.

Note that this is decidedly in contrast to the modal principle Hellman cites in [4] to ensure his potentialist set theory satisfies replacement. The principle Hellman cites (Strong EP[11]) simply baldly asserts that when it is necessary to collect the image of the set a under a definable function to satisfy replacement then its possible to extend M_0 (the initial segment containing a) to an initial segment M doing just that. As this merely restates the demands of replacement in a potentialist framework (while requiring belief in the logical coherence of strong large cardinal assumptions), it lacks any extra intuitive

[11]Hellman's Strong EP principle reads as follows.

> Let $\phi(x, y)$ be a formula "defining a function", where this is spelled out by writing out the Putnam translate of the usual condition; further let a be any set in any full model such that, for any x in a, M_β is the least full model containing the unique y such that $\phi(x, y)$. Then it is possible that there exists a common proper extension, M, of all such M_β.

force and claims about extensions of a technically complex notion of initial segment of the sets hardly qualifies as straightforward or obvious[12][13].

4. A More Cautious Formulation of Potentialist Set Theory

First let me lay out our preferred formulation of potentialist set theory.

4.1. **A Note about Logic.** Let me begin with a note about some background logical assumptions we will be making.

In what follows we will appeal to a primitive (proof transcendent[14]) notion of logical possibility. Note that this notion of logical possibility isn't just some arbitrary multiplication of modal primitives but is implicit in our notion of logical validity. If, as Hartry Field advocates in [3], you accept a primitive notion of logical validity then Ψ is logically necessary just if the inference from empty premises to Ψ is logically valid. Note that by iterating this process we can coherently talk about the logical possibility not only of statements in our base logic but also of statements themselves phrased in terms of logical possibility.

In this paper we will adopt the formal system for reasoning about the logical possibility of statements in second order logic from sections 4.2.1 and 4.2.2 of [14] which gives an explicit precisification of the system used by Hellman in [4] as a positive free logic[15] with second order relational quantification

[12]One might be tempted to think that the difficulties with Hellman's Strong EP principle are merely notational and that if we just avoided formulating it in terms of initial segment extensions we'd recover the kind of intuitive, simple and compelling principle we introduce latter. However, closer examination reveals such a transformation is anything but easy. For example, Hellman's system takes initial segments to be full models of ZF_2 meaning that Strong EP is implicitly asking us to accept some highly non-obvious claims about the coherence of large cardinal axioms. For another, the casual presentation of this principle hides its true complexity, e.g., the statement M_β is a common proper extension must be spelled out in some syntactically appropriate fashion.

[13]Like Hellman, Linnebo also proposes a principle which lets one justify his potentialist version of the axiom of replacement in [7]. However, he too makes little claim for the obviousness of the principle he picks, and his justification of replacement is prima facie tied up with the distinctive philosophical commitments of his Parsonian approach to set theory.

[14]That is, unlike first order logic, facts about logical possibility outrun what we can prove (in any sound system whose consequences are computably enumerable) about them.

[15]We will regard the logic to only contain the logical symbols $\exists, \Diamond, \wedge, \neg$ with the remaining propositional, quantificatorial and modal notions defined in terms of these in the usual way.

with both first and second order equality[16] satisfying S5. As is standard we will let capital letters denote second order relational variables, the lower case letters f, g, h, k denote functions[17] and other lower case letters first order variables and, when necessary, use superscripts to denote the arity of a relational variable. This system formalizes reasoning about the logical possibility of statements in second order logic where each possible scenario (or 'world') can be thought of as specifying a domain of objects second order quantifiers can range over under the assumption that a second order relation R exists in a particular scenario iff every x_i in a tuple $x_1, \dots x_n$ satisfying R exists at that world[18]. We will call the collection of statements expressible in this system the language of logical possibility. Note that our use of these parts of the formal system in [14] does not extend to either the axioms regarding ordered pairs nor imply assent to the interpretation of second order quantifiers as pluralities.

It is worth noting that unlike Hellman this system allows (and we will make use of) second order comprehension over modal formulas[19] (though again only ones in which all 'quantified-in' objects must exist).

[16]It is common in many treatments of second order logic to replace the explicit logical relation of equality with a definition in terms of set membership, e.g., $x = y$ if x and y belong to all the same sets and $X = Y$ if they have all the same members. However, the ability to mention terms which don't exist in the current scenario in positive free logic prevents us from taking this route.

[17]As is common we will understand quantification over functions as quantification over binary relations satisfying the requirement that

$$(\forall x)\,[(\exists y)R(x,y) \implies (\exists! y)R(x,y)]$$

[18]Formally, this is expressed by the principles PL1 and PL2 from [14]

$$x \in X \rightarrow \Box\,(\mathbf{E}(X) \implies \mathbf{E}(x) \wedge x \in X) \tag{PL1}$$

$$\Box(\forall x)\,[\Diamond(x \in X) \iff \Diamond(x \in Y)] \implies X = Y \tag{PL2}$$

This amounts to thinking of second order objects extensionally rather than rather than intentionally.

[19]While someone considering this system from a purely formal point of view might worry that allowing such a powerful form of comprehension might allow for some kind of paradoxical self-reference it can't raise any new worries regarding the philosophical acceptability of these concepts. If one accepts the coherence of the kind of quantified-in statements we consider then there is a genuine matter of fact whether a formula $\phi(x)$ expressed in this system holds for any particular object x. As accepting true second order logic (rather than a mere first-order masquerade) means believing that absolutely any way of choosing objects gives rise to a second order set if using comprehension with respect to $\phi(x)$ leads to a contradiction then our system was already incoherent.

Recent work by REDACTED suggests that we can replace quantifying-in by introducing the notion of logical possibility holding certain structural facts fixed[20].

4.2. **Notation.** With this background logic in mind we will now introduce some notation for frequently used expressions in the remainder of this paper.

Notation 4.1.

$$\mathbf{E}(x) \overset{\text{def}}{\iff} (\exists x')(x' = x)$$
$$\mathbf{E}(X) \overset{\text{def}}{\iff} (\exists X')(X' = X)$$

- $\Diamond_{x_1,..x_n,R_1..R_m} \Psi(x_1, ..x_n, R_1..R_m) \overset{\text{def}}{\iff} \Diamond\, \mathbf{E}(x_1) \wedge \ldots \mathbf{E}(x_m) \wedge E(R_1) .. \wedge E(R_m) \wedge \Psi(x_1, ..x_n, R_1..R_m)$, i.e. it is logically possible that $x_1, ..x_n, R_1..R_m$ exist, and Φ holds true of them
- (and analogously) $\Box_{x_1,..x_n,R_1..R_m} \overset{\text{def}}{\iff} \Box E(a_1) \wedge ..E(a_m) \wedge E(R_1) .. \wedge E(R_m) \rightarrow \Psi(x_1, ..x_n, R_1..R_m)$. i.e., that it's logically necessary that if $x_1, ..x_n, R_1..R_m$ exist then $\Psi(x_1, ..x_n, R_1..R_m)$

Notation 4.2.

$$(\forall x \mid \psi(x))(\phi(x)) \overset{\text{def}}{\iff} (\forall x)(\psi(x) \implies \phi(x))$$
$$(\exists x \mid \psi(x))(\phi(x)) \overset{\text{def}}{\iff} (\exists x)(\psi(x) \wedge \phi(x))$$
$$\phi \subset \psi \overset{\text{def}}{\iff} (\forall x)\,(\phi(x) \implies \psi(x))$$

Note that the final convention here allows the use of the $\subset$ symbol both for second order variables and explicit formulas.

As it will frequently be necessary to refer to the collection of objects related by some second order relations $R_1, \ldots R_n$ apply to (in the sense that we might say, e.g. 'the numbers are the objects which are related by the relations $+, *, S$') we also introduce the following notation.

Notation 4.3. Let $\text{Ext}(R_1, \ldots R_n)(y)$ abbreviate the claim that y appears in some tuple $\vec{v}$ where $R_i(\vec{v})$ for some i. More formally, $\text{Ext}(R_1, \ldots R_n)(y)$ abbreviates the disjunction of

$$(\exists x_1) \ldots, (\exists x_{j-1}), (\exists x_{j+1}), \ldots, (\exists x_{l_i}) R_i(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_{l_i})$$

for $1 \leq i \leq n$ and $1 \leq j \leq l_i$ where l_i is the arity of R_i.

[20]From a formal perspective this approach is identical to allowing quantifying-in to contexts in which the quantified-in second order objects exist. However, from a philosophical perspective we can understand this notion in terms of the logical possibility of scenarios containing 'isomorphic' structures rather than *de re* identity across possible worlds and thereby avoid the philosophical concerns with this issue.

4.3. **Characterizing Initial Segments.** With these points about logic in place, let us now turn to articulating potentialist set theory. The first step is to cash out the idea of being an initial segment invoked above.

Hellman's influential formulation of potentialism in [4] requires that initial segments satisfy ZF^2. However, giving such a strong definition of an initial segment builds in the assumption that it is coherent to suppose there is a definite totality satisfying ZF^2 including replacement.

As we wish to derive replacement from a unified conception of the sets, we must be more careful in our definition of an initial segment. In particular, we take the definition of an initial segment to be exactly that given by the idea of sets forming a cumulative hierarchy discussed above[21], including following Boolos in thinking of iterative heirarchies as containing two kinds of first order objects: sets and stages[2]. Thus some objects (considered under a second order relation $\in, <$ and @) will qualify as an initial segment iff they consist of some levels (well ordered by $<$), plus levels of sets, with each each level growing by including sets corresponding to all possible ways of choosing sets available at an earlier level[22].

First, this requires we define just what constitutes a stage.

Definition 4.4 (Well-ordering)**.** The relation $<$ is a well-ordering if it is a linear order[23] and

$$(\forall X \mid X \subset \mathrm{Ext}(<))$$
$$(\exists y)(X(y)) \to (\exists z : X(z)(\forall w : X(w))(z \leq w)$$

where $z \leq w$ abbreviates $z = w \vee z < w$

Note that this definition requires that all well-orders have height at least 2 but as we won't need to make use of well-orders of height 0 or 1 this will not matter.

We can now define the property of being built up at level α of the cumulative hierarchy. In particular we define the property of the tuple $V = (\in, <, @)$

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[21]Modulo the fact that, for technical reasons, we will only consider initial segments of height at least 2.

[22]Note that this definition follows Boolos (and thereby differs from standard presentations like [5] in a very minor way) concerning limit ordinals. On the usual definition when α is a limit, V_α is just the union of V_β with $\beta < \alpha$. On this definition, V_α also includes the all possible subsets of $\cup_{\beta<\alpha} V_\beta$. Thus, for α a limit, our definition of V_α corresponds to $V_{\alpha+1}$ on the standard definition.

[23]$<$ is a linear order if for all x, y, z in Ext($<$) the following hold

- Totality $x = y \vee x < y \vee y < x$
- Irreflexivity $\neg(x < x)$
- Transitivity $x < y \wedge y < z \implies x < z$

being a witnessed initial segment of height $<$ (which for technical reasons we will require to be at least 2).

First, however, we introduce the following notation

Notation 4.5. Given $V = (\in, <, @)$ we introduce the following notation.

- $\text{set}(x) \overset{\text{def}}{\iff} (\exists o \mid \text{ord}(o))(@(x, o))$
- $\text{ord}(o) \overset{\text{def}}{\iff} \text{Ext}(<)(o)$
- $V(x) \overset{\text{def}}{\iff} \text{set}(x) \vee \text{ord}(x)$

We call those objects satisfying $\text{set}(x)$ sets and those satisfying $\text{ord}(o)$ ordinals (note that this differs from standard set theoretic terminology in not requiring that ordinals be sets).

Definition 4.6. The tuple $V = (\in, <, @)$ where $<$, $\in$ and $@$ are binary relations is an *initial segment* just if

(1) $<$ is a well-ordering and $\exists y\, \text{ord}(y)$.

(2) $(\forall x \mid V(x))\, (\forall o \mid V(o))\, (@(x, o) \implies \text{set}(x) \wedge \text{ord}(o))$

(3) $(\forall x \mid \text{set}(x))(\exists o \mid \text{ord}(o))@(x, o)$

(4) Every subset of levels below o appears as a set at level o (Fatness)

$$\begin{aligned}&(\forall o \mid \text{ord}(o)(\forall P)\\ &\quad (\forall x \mid P(x)(\exists o' < o)(@(x, o'))\\ &\implies (\exists y)(@(y, o) \wedge (\forall z)(P(z) \leftrightarrow z \in y))]\end{aligned}$$

(5) Only sets built up in the above way are allowed (thinness).

$$(\forall o)(\forall x \mid @(x, o))(\forall z \mid z \in x)(\exists o' < o)(@(z, o')))$$

(6) Extensionality

$$(\forall x \mid \text{set}(x))(\forall y \mid \text{set}(y))\, (x = y \iff [(\forall z)(z \in x \iff z \in y)])$$

(7) $(\forall x)\neg(\text{ord}(x) \wedge \text{set}(x))$.

We abbreviate the conjunction of the above with the claim that $\mathbf{E}(\in) \wedge \mathbf{E}(<) \wedge \mathbf{E}(@)$ as $\mathscr{V}(V)$.

We must also say what it takes one initial segment $V' = (\in', <', @')$ to extend some other initial segment $V = (\in, <, @)$. Intuitively this occurs just when the relations $(\in', <', @')$ agree with $(\in, <, @)$ wherever the former are defined and don't add any ordinals/sets below the ordinals/sets in V, i.e., V' is an end-extension of V. Formally, we can express this as follows.

Definition 4.7. $V' = (\in', <', @')$ extends $V = (\in, <, @)$ (denoted by $V \leq V'$) just if the following conditions are all satisfied

(1) $\mathscr{V}(V) \wedge \mathscr{V}(V')$

(2) $(\forall x \mid \text{set}'(x))(\forall o \mid \text{ord}(o))(@(x,o) \iff @'(x,o))$

(3) $(\forall y \mid \text{set}(y))(\forall z)(z \in' y \iff z \in y)$

(4) $(\forall o \mid \text{ord}(o))(\forall u)(u <' o \iff u < o)$

If $V \leq V'$ but $\text{ord}' \not\subseteq \text{ord}$ (or equivalently $\text{set}' \not\subseteq \text{set}$) we write $V < V'$

Note that condition 4 implies that every ordinal in V is an ordinal in V' as the ordinals in V are just those objects in $\text{Ext}(<)$ and those in V' are those objects in $\text{Ext}(<')$.

4.4. **Formalizing the Potentialist Translation.** So we can write up our proposed variant on Hellman's potentialist set theory as follows.

Definition 4.8 (Potentialistic Truth)**.** For $V = \in, @, <$ an initial segment and ϕ a formula in the language of set theory with parameters from the sets in V[24] we define a formula $V \models_p \phi$ in the language of logical possibility via recursion the structure of ϕ as follows (where $\Diamond_V$ abbreviates $\Diamond_{\in,<,@}$ and a, b are sets in V).

$$\begin{aligned}
V \models_p a \in b &\iff a \in b \\
V \models_p a = b &\iff a = b \\
V \models_p \neg\phi &\iff \neg V \models_p \phi \\
V \models_p \phi \wedge \psi &\iff V \models_p \phi \wedge V \models_p \psi \\
V \models_p \exists x \phi(x) &\iff \Diamond_V \left(\exists V'\right) \left(\exists y \mid \text{set}'(y)\right) \left[V' \geq V \wedge V' \models_p \phi(y)\right]
\end{aligned}$$

We define the *potentialist translation of* a set theoretic *sentence* ϕ (without parameters) to be $\Box\left[(\forall V)\mathscr{V}(V) \implies V \models_p \phi\right]$[25] .

It is worth noting that this definition corresponds to the following rule in the case of universal quantification.

$$V \models_p \forall a \phi(a) \iff \Box_V (\exists V')(\exists a)[V' \geq V \wedge set'(a) \to V' \models_p \phi(a)]$$

[24]Note that this is equivalent to augmenting formulas with free variables with an interpretation assigning those free variables to sets in V.

[25]By corollary B.1 this is not a trivial notion, i.e. initial segments are logically possible.

In what follows we will make use of several lemmas. The interested reader may consult the appendix for detailed proofs. The definition of isomorphism of initial segments is exactly what one would expect but see definition B.2 in the appendix for a formal statement.

seem forced to

Lemma 4.9 (Isomorphism Agreement)**.** *Given initial segments V, $\widehat{V}$ if h, k are both isomorphisms from V to $\widehat{V}$ then $h = k$, i.e, $(\forall x \mid V(x))(h(x) = k(x))$.*

Furthermore, suppose $V_f \leq V, V_g \leq V$ and $V'_f \leq V', V'_g \leq V'$. If f is an isomorphism of V_f with V'_f and g and isomorphism of V_g with V'_g then $f(x)$ is equal to $g(x)$ on any x in both V_f and V_g

Lemma 4.10 (Comparibility)**.** *Given initial segments V, V' either V is isomorphic to an initial segment $\widehat{V} \leq V'$ or V' is isomorphic to an initial segment $\widehat{V} \leq V$.*

Lemma 4.11 (Stability Lemma)**.** *If $V' \geq V$ and all parameters in ϕ are sets in V then*

$$V \models_p \phi \iff V' \models_p \phi$$

.

Together these principles imply that a sentence ϕ (having no free variables) will have the same potentalist truth value relative to all initial segments.

Lemma 4.12 (Bounded Quantifiers Lemma)**.** *If V is an initial segment then*

$$[V \models_p (\forall x)(x \in a \rightarrow \phi(x, a))] \iff [(\forall x)(x \in a \rightarrow V \models_p \phi(x, a))]$$

4.5. **A Principle of Extendability.** For this notion of potentalistic truth to vindicate the power set axiom (or even pairing), we must draw on the 'indefinite extendability' intuition above, to get a principle ensuring that it is always logically possible to extend any initial segment.

First we need to introduce some terminology.

Definition 4.13 (Layer Of Classes)**.** The pair $C, \in^*$ form *a layer of classes over* $\text{Ext}(R_1, \ldots R_n)$ just if all of the following hold.

- The relation $\in^*$ relates elements in $\text{Ext}(R_1, \ldots R_n)$ to classes.
$$(\forall x)(\forall y)(x \in y \implies \text{Ext}(R_1, \ldots R_n)(x) \wedge C(y)$$
- Classes are disjoint from their elements.
$$(\forall x \mid C(x))(\neg \text{Ext}(R_1, \ldots R_n)(x))$$

- Every subset of $\mathrm{Ext}(R_1, \dots R_n)$ is a class.
  $(\forall X \mid X \subset \mathrm{Ext}(R_1, \dots R_n))\, (\exists y \mid C(y)(\forall x)(x \in^* y \leftrightarrow X(x)))$
- Extensionality
  $(\forall x \mid C(x))\, (\forall y \mid C(y))\, [x = y \iff (\forall z)\, (z \in^* x \iff z \in^* y)]$

Let $\mathscr{C}(C, \in^*, \mathrm{Ext}(R_1, \dots R_n))$ abbreviate the claim that the conjunction of the above claims hold with respect to $C, \in^*$ and $\mathrm{Ext}(R_1, \dots R_n)$.

Given any logically possible scenario, we take it as an intuitive principle that augmenting that scenario with a layer of classes doesn't violate any law of logic. While one could dispute this claim anyone inclined to accept set theory in the first place (especially anyone who feels the attractions of potentialist set theory as described above) is unlikely to be so inclined. Formalizing this intuition gives us the following principle.

Principle 4.14 (Superset Extendability)**.**

$$\Box \forall R_1, \dots, R_n \Diamond_{R_1, \dots, R_n} \left[(\exists C, \in^*) \mathscr{C} \left(C, \in^*, \mathrm{Ext}(R_1, \dots R_n) \right) \right].$$

Note that while it might seem odd that we've singled out the relations of arity 1 for special treatment the above principle is sufficient to allow us to augment the objects satisfying $\mathrm{Ext}(R_1, \dots, R_n)$ with objects coding all relations of arity n for any natural number n[26]

Intuitively, this principle should be enough to show that it is logically possible to properly extend any initial segment by supplementing the existing sets with those classes from the above principle which don't already form sets. The interested reader may find the details of the proof of the following lemma in the appendix.

Lemma 4.15 (Proper Extension Lemma)**.**

$$\Box (\forall V) \left[\mathscr{V}(V) \implies \Diamond_V (\exists V')(V' > V) \right]$$

5. The Hedgehog

Now it's time to introduce the key modal principle which We will use to justify the Axioms Schema of Replacement (from a potentialist point of view). We call this principle the Hedgehog for reasons that will be illustrated below.

[26]By applying the principle $n + 1$ times and repeatedly using the standard set theoretic definition of a pair (understood in terms of classes) $\{x, \{y\}\}$ we can supplement the existing objects with n tuples of those objects. Then, by applying the principle once more, we can deduce the possibility of a layer of classes of n-tuples allowing us to deduce the logical possibility of augmenting the existing objects with objects coding all the n-ary relations over our initial domain with first-order objects.

Very crudely, the Hedgehog principle is a 'simultaneous realizability' principle. It extends the idea that for any two 'logically possible scenarios' α and β, there is nothing logically incoherent about supposing we have the objects (and relations) from scenario α and β existing side-by-side.

More generally, it expresses the idea that if it is logically coherent to have a collection I of objects and, for each object i satisfying I there is a logically possible scenario S_i, then the scenario corresponding to the disjoint union[27] over I of the scenarios S_i should also be logically possible. After all, there was nothing logically incoherent about each S_i individually, nor in having a collection I, so the Hedgehog principle expresses the intuition that there are no surprise constraints on logical coherence which prevent this disjoint union from being logically possible.

Of course, the language of logical possibility (by design) doesn't let us directly express the above claim about disjoint unions. Instead, we must try and capture a portion of this intuition syntactically. In particular, we will attempt to capture the idea that if, for every i satisfying I, it is logically possible that some formula $\Psi(i)$ obtains, then it is logically possible to have a collection of domains D_i where D_i witnesses the logical possibility of $\Psi(i)$. Note that, while the initial intuition might suggest that the domains D_i should be disjoint, we instead allow them to all extend (the objects mentioned by) a common list of parameters – so that Ψ can use these parameters in expressing the possibility statement $\Psi(i)$.

The Hedgehog Schema gets its name from the fact that it (metaphorically) says the following. If, for each object of a certain kind 'each pore' within a certain structure 'the hedgehog's body', we can specify a certain kind of logically possible expansion of the hedgehog's body 'a quill', then the original structure could exist while all objects simultaniously have such (disjoint) extensions. In this metaphor the pores are the objects in some index collection of objects I, and the Hedgehog's body is the collection of objects related by some structure of relations $\langle O_1 ... O_n, I\rangle$. The quill belonging to a pore x is a collection of objects such that a formula $\phi(x)$ is satisfied within the domain you get by adding that quill to the Hedgehog's body. Thus an instance Hedgehog Schema says that if each pore can extrude a quill, then you can have a full Hedgehog where every pores has a separate quill growing out of it.

To state the Hedgehog principle more formally, we will want to consider what is true relative to some restricted domain indexed by members of some collection I. We associate each $i \in I$ with a domain D_i determined by $D_i = \{x \mid D(i,x)\}$ (or alternatively $D_i(x) \iff D(i,x)$) and now define $D_i \models_D \phi$.

[27]That is we imagine swapping out elements of each S_i until they are all disjoint and then taking their union.

Definition 5.1. We define $D_i \models_D \phi$ by induction on formula complexity.

$$\begin{aligned}
D_i \models_D \phi \wedge \psi &\iff D_i \models_D \phi \wedge D_i \models_D \psi \\
D_i \models_D \neg\phi &\iff \neg D_i \models_D \phi \\
D_i \models_D \exists x \phi(x) &\iff (\exists x \mid D(i,x))(D_i \models_D \phi(x)) \\
D_i \models_D \exists R \phi(R) &\iff (\exists R \mid \mathrm{Ext}(R) \subseteq D_i)(D_i \models_D \phi(R)) \\
D_i \models_D \Diamond\phi &\iff \Diamond\phi
\end{aligned}$$

The notation $D_i \models_D \phi$ is justified by the fact that $D_i \models_D \phi$ holds just ϕ is modeled by the structure whose domain is D_i and the relations appearing in ϕ are interpreted to be the restriction of those relations to D_i. For example, suppose for i a natural number $D_i = \{x \mid x < i\}$ (so $D_2 = \{0, 1\}$). Then $D_1 \models \forall x \forall y(x = y)$ but $D_2 \models \exists x \exists y(x \neq y)$.

Note that the following lemma follows straightforwardly[28] from the definition of $\models_D$ and $\models_p$.

Lemma 5.2 (Restricted Content Lemma)**.** *If V is an initial segment and $D \subset V$ then*

$$V \models_p \phi \iff D \models_D [V \models_p \phi]$$

We are now in a position to state the Hedgehog principle formally. Despite the intimidating looking formalism it is important to remember it is expressing the simple intuition described above.

Principle 5.3 (Hedgehog)**.** *If $\Psi(i, O_1 \ldots, O_n, I)$ is a formula in the language of logical possibility with only $i, O_1, \ldots, O_n, I$ as free variables then:*

$$\begin{aligned}
&\forall O_1, \ldots, O_n \forall I \\
&\quad \big[(\forall i \mid I(i))\Diamond_{O_1,\ldots,O_n,I}\Psi(i, O_1, \ldots, O_n, I)\big] \implies \Diamond_{O_1,\ldots,O_n,I}(\exists D)(\forall i \mid \underline{I}(i))\Big[\\
&\qquad (\forall j \mid I(j))(\forall x)[D(i,x) \wedge D(j,x) \iff (i = j \vee \mathrm{Ext}(O_1, \ldots, O_n, I)(x)] \\
&\qquad \wedge D_i \models_D \Psi(i, O_1, \ldots, O_n, I)\Big]
\end{aligned}$$

As an informal example suppose, for each i in some domain $\mathbb{N}$ with the structure of the natural numbers it is logically possible to have a group G_i (disjoint from $\mathbb{N}$) of size 2^i then the Hedgehog principle would let us conclude that it was logically possible to simultaneously realize $\mathbb{N}$ and (disjoint copies of) the G_i.

[28]A fully formal proof would proceed by induction on formula complexity but, like many results about formal syntax, this simply gives a formal gloss to the idea that $\Psi \iff D \models_D \Psi$ whenever all relations and quantification in Ψ are restricted to D.

From a purely formal point of view, this principle is (by design) quite similar to the axiom schema of replacement. Thus, one might naturally wonder how we can assert that the Hedgehog is a natural consequence of an intuitively consistent concept of logical possibility while denying that the iterative hierarchy conception of the sets closed under replacement constitutes a similarly intuitively consistent concept. The difference is the axiom of replacement asks us to accept there is a logically coherent concept, set, which applies to a collection of objects which is closed under definable functions. As discussed above nothing about the iterative hierarchy conception of the sets lets us see that such a collection should exist. Indeed, there are many examples of incoherent closure conditions we would like the sets to obey, e.g., every collection of sets is a set, which they obviously can't.

In contrast, the intuition motivating the Hedgehog principle doesn't ask us to accept the logical coherence of any totality of 'scenarios' closed under the operation of disjoint union, but merely that whenever some number of individually specified scenarios S_i are all individually logically coherent, so too is the scenario reflecting the disjoint union of these scenarios. So, unlike replacement, the Hedgehog is merely a rule for building a new scenario from an existing one.

The actualist says that it's possible for there to be a single structure that's closed under this condition; the potentialist says that, for any possible structure, it's possible to extend it as specified by this condition. If one could coherently imagine a single structure that witnessed all logical possibilities, this would be the same – but obviously this is what the potentialist wants to deny.

5.1. **Second Order Choice Principle.** One final component is needed before we can explain our justification of replacement. For, unfortunately, the Hedgehog principle only lets us infer the simultaneous logical possibility of *bare domains* D_i of some specified cardinality. But we want to be able to assert the simultaneous possibility of *structures* such as initial segments of the sets. So we need a principle which lets us go from the fact that, for each i, it is possible to impose a certain structure $\in_i$ on D_i to a the possibility of a single indexed relation $\in_\Sigma$ which picks a witnessing relation $\in_i$ D_i, and then codes up and stores the behavior of this relation.

Principle 5.4 (Second Order Choice)**.**

$$(\forall i : I(i))(\exists R_i)\phi[i, R_i]) \rightarrow (\exists R^*)(\forall i : I(i))\phi[i, R^*(i, \cdot)]$$

Where each R_i has the same arity and $\phi[i, R^*(i, \cdot)]$ abbreviates the claim that

$$\exists Q \forall \vec{x} Q(\vec{x}) \iff R^*(i, \vec{x}) \land \phi[i, Q]$$

The same intuitions that motivate the familiar Axiom of Choice in set theory, motivate this second order choice principle. If, for each $i \mid I(i)$, there is at least one second order relation R_i satisfying some property $\phi(i, R_i)$, then it should be possible to 'choose' a way of pairing each i with some suitable R_i. So there should be a second order relation R^* which witnesses this fact (by coding up the behavior of the R_i associated with each i).

Importantly, it is possible to derive the uses we will make of second order choice from the Superset Extendability and the standard formulation of choice such as the one Shapiro accepts as part of his system of second order logic in [15][29].

6. Justifying Replacement

Now let's turn to justifying the Replacement Schema in ZFC from a potentialist point of view. We will use an instance of the Hedgehog schema above (plus other intuitive reasoning), to show that the potentialist translation of the Replacement axioms are true. More formally, I'll prove the following theorem.

Theorem 6.1 (Potentialist Replacement)**.** *Necessarily, If V is an initial segment and φ is an instance of the replacement schema given in equation Repl then $V \models_p \varphi$.*

To prove this theorem it is enough to show that, for any formula ϕ in the language of set theory and any initial segment V_0 and parameters $w_1, \dots w_n$ and i_0 in V_0 the following conditional holds with logical necessity. If $V_0 \models_p (\forall i)(i \in i_0 \to \exists! y\, \phi(i, y))$ then it is logically possible for there to be an initial segment $\widehat{V_\Omega} \geq V_0$ and j such that $\widehat{\text{set}_\Omega}(j)$ such that $V_\Omega \models_p \forall i\big(i \in i_0 \to$

[29]Specifically we can derive

$$(\forall i : I(i))(\exists R_i)\phi[i, R_i]) \to \Diamond(\exists R^*)(\forall i : I(i))\phi[i, R^*(i, \cdot)]$$

. using the following formulation of choice (equivalent to the one in [15] but phrased in terms of relations instead of functions)

$$(\forall R)(\exists R')(\forall x_1, \dots, x_n)\left[(\exists y)R(x_1, \dots, x_n, y) \implies (\exists! y)\left(R'(x_1, \dots, x_n, y) \wedge R(x_1, \dots, x_n, y)\right)\right]$$

The derivation proceeds by using the layer of classes principle (as outlined in footnote 26) to deduce the possibility of having first order objects corresponding to every n place relation. We then use comprehension to deduce the existence of a relation $W(i, y)$ which holds just if $I(i)$ y codes the n place relation R_i and $\phi[i, R_i]$. We then apply the choice principle to deduce the existence of a W' such that for each i in I there is a unique y such that $W'(i, y)$. Using comprehension we define $R^*(i, x_1, \dots, x_n)$ to hold just when there is some y such that $W'(i, y)$, y codes for R_i and $R_i(x_1, \dots, x_n)$.
This result has the odd property that it establishes the logical possibility of the existence of an n place relation R^* but not the actual existence of this relation despite the fact that $\text{Ext}\, R^*$ is contained in the actual world. To avoid this bizarre outcome in a full treatment one might wish to augment the rules of this system with the following principle for every formula ϕ and natural number n.

$$(\forall Y)\Diamond\left[(\exists R^n)(\phi(R^n) \wedge \text{Ext}(R^n) \subseteq Y)\right] \implies (\exists R^n)(\phi(R^n) \wedge \text{Ext}(R^n) \subseteq Y)$$

$\exists y(y \in j \wedge \phi(i,y)\big)$. The sufficiency of this claim follows from the Stability Lemma and the definition of $\models_p$.

Our strategy will be to first demonstrate that whenever the antecedent holds, then for any $i \in i_0$ it is possible to have an initial segment V_i extending V_0 and an element y in V_i such that $V_i \models_p \phi(i,y)$. If we could simply define $V_\Sigma = \bigcup_{i \in i_0} V_i$ (where we assume the initial segments in this union have been rendered compatible) and then consider the initial segment $\widehat{V}_\Omega$ produced by applying the Proper Extension Lemma to $V_\sim$ we would be done. After all, for every $i \in i_0$, $\widehat{V}_\Omega$ contains a y such that $V_i \models_p \phi(i,y)$ (and hence $V_\Omega \models_p \phi(i,y)$ by the stability lemma). Moreover, by fatness, the Stability Lemma and comprehension $\widehat{V}_\Omega$ contains a set j consisting of those y such that $\exists i \in_\omega i_0 \widehat{V}_\Omega \models_p \phi(i,y)$, giving us the desired conclusion[30].

However, we can't take unions over possible initial segments, as these initial segments live in different logically possible 'scenarios.' To overcome this difficulty, we will invoke the Hedgehog principle to derive that there could be a single scenario containing an initial segment V_i as above for each $i \in_0 i_0$. In particular, letting I be such that $(\forall i)(I(i) \leftrightarrow i \in_0 i_0)$ we use an instance the Hedgehog Schema to infer the logical possibility of a single scenario containing V_0, I and, for every i satisfying I, a domain D_i containing an initial segment $V_i \geq V_0$ such that $V_i \models_p \phi(i,y)$.

We will then argue for the logical possibility of a single structure $V_\sim$ which extends an isomorphic copy, $\widehat{V}_i$, for each V_i. Let, $[x]$ denotes the isomorphic image of x from V_i in $\widehat{V}_i$. As, for each i with $i \in_0 i_0$, we have a V_i and b such that $V_i \models_p \phi(i,b)$ we can conclude that for each such i there is a $\widehat{V}_i \leq V_\sim$ such that $\widehat{V}_i \models_p \phi([i],[b])$, and therefore $V_\sim \models_p \phi([i],[b])$.

Replacing $V_\sim$ with an isomorphic copy extending V_0 (see lemma B.6) we then employ the Proper Extension Lemma to build V_Ω by adding a further layer of sets. Now, by construction, V_Ω contains a j collecting all the y in $V_\sim$ such that $(\exists i)(I(i) \wedge V_\sim \models_p \phi(i,y))$. So, for all $i \in_0 i_0$, there is some $y \in_\Omega j$ such that $V_\Omega \models_p \phi(i,y))$. By the definition of $\models_p$, the definition of extends and the Bounded Quantifiers Lemma)it follows that $V_\Omega \models_p (\forall i)(i \in i_0 \rightarrow (\exists y)(y \in j \wedge \phi(i,y))$.

To flesh out this proof, we will now give details about the application of the Hedgehog, and the construction of V_Ω.

6.1. **Part I: Deploying the Hedgehog.** As described above, the goal in applying the Hedgehog is to demonstrate (from the assumption above) the simultaneous logical possibility of initial segments V_i (for each i in I)

[30]By the Stability Lemma it is enough to show that we can collect together the witnesses to $V_i \models_p \phi(i,y)$ for $i \in_0 i_0$ in a single set j even though, were we attempting to construct a normal actualist model, we couldn't assume that $V_i \models \phi(i,y)$ implies that $\phi(i,y)$.

containing a y such that $V_i \models_p \phi(i, y)$. We thus apply the The Hedgehog principle to the following formula

$$\Psi(i, V_0, I) \overset{\text{def}}{\iff} (\exists V_i \geq V_0)\,(\exists y \mid \mathrm{set}_i(y))\,(V_i \models_p \phi(i, y))$$

where the variable O_1 in the statement of the The Hedgehog is played by V_0 and I is the second order object (guaranteed to exist by comprehension) containing just those i satisfying $i \in_0 i_0$.

On the assumption that the antecedent of replacement holds, it is straightforward to verify that, for every i in I, $\Diamond_{V_0} \Psi(i, V_0, I)$ holds simply by unpacking the definition of $\models_p$ to yield V_i containing y and using the fact that $\mathrm{set}_0(x)$ and $\mathrm{set}_0(i_0)$. Using the The Hedgehog principle we can now infer that it is possible to have an indexed collection of domains D_i, such that for each i such that $I(i)$, $D_i \models_D \Psi(i, V_0, I)$ and that D_x and D_y for $x \neq y$ are disjoint except for those elements a satisfying $V_0(a)$.

Applying the definition of $\models_D$ to the first $\exists V_i$ quantifier and then applying Second Order Choice, we can infer the existence of an indexed collection V_i (technically[31] three place relations $\in (i, a, b), < (i, o, o'), @(i, a, o))$ such that V_i witnesses the truth of $\Psi(i, V_0, I)$ where V_i, $V_{i'}$ $(i \neq i')$ both extend V_0 and are otherwise disjoint.

In the next section we will show that this disjoint (over V_0) union can be used to construct a single initial segment which extends V_0 and 'combines' (isomorphic copies of) every V_i in the disjoint union.

6.2. Part II: Constructing a Suitable V_Ω.

As outlined above, our strategy will be to build $V_\sim$ (which 'combines' all the V_is) out of equivalence classes. So our first step is to establish the logical possibility of further supplementing the structure given by the indexed relations $\in, <, @$ with additional objects that can function as equivalence classes.

We do this by applying the Superset Extendability principle to infer the logical possibility of a scenario in which the relations $\in, <, @$ are supplemented by a relation $\in^*$ coding a layer of classes over $\mathrm{Ext}(D)$. Technically, this means we are demonstrating something of the form $\Diamond\Diamond\exists V_\omega \geq V_0 \ldots$ but as S5 allows us to infer $\Diamond\phi$ from $\Diamond\Diamond\phi$ we need not track the number of logical possibility operators.

[31]Formally, we use second order to choice to establish the existance of three places relations $\in, <, @$ that for each i in I there are relations $(\in_i, <_i, @_i$ such that

$$(\forall a, b)\,(\in (a, b, i) \iff a \in_i b)$$
$$(\forall u, o)\,(< (u, o, i) \iff u \in_i o)$$
$$(\forall o, x)\,(@(x, o, i) \iff @_i(o, x))$$

Now that we have this extra layer of classes over $\mathrm{Ext}(D)$, we must specify what equivalence classes to consider. Intuitively speaking, we will want to 'identify' elements of V_i and $V_{i'}$, iff they play identical roles. More formally,

Definition 6.2. Define $a \sim b$ just if there exists i, j in I, initial segments $V_i^* \leq V_i$ and $V_j^* \leq V_j$ and an isomorphism f of V_i^* with V_j^* with $f(a) = b$

As suggested by our notation (and the idea that $\sim$ identifies objects which 'play identical roles'), $\sim$ is an equivalence relation (see lemma C.1 for details). To construct $V_\sim$ we will need to choose objects to play the role of the equivalence classes induced by $\sim$. As we've supplemented the original scenario (provided by the Hedgehog) with a layer of classes, we can identify the equivalence classes induced by $\sim$ with the class (a first order object) containing just the elements in that equivalence class.

So, for each a such that $(\exists i \mid I(i))\,(V_i(a))$, there is a class, call it $[a]$, which contains exactly the b such that $a \sim b$ (i.e., exactly the b that 'play the same role' as a within some V_j'). Finally we define $V_\sim$ on the above equivalence classes, as follows.

Definition 6.3. Let $V_\sim$ be the structure given by $V_\sim = (\in_\sim, @_\sim, \leq_\sim)$ where these relations are given by (applying comprehension to) the following formulas.

$$\begin{aligned}
z \in_\sim y &\iff (\exists i \mid I(i))(\exists z_i, y_i)(\mathrm{set}_i(z_i) \wedge \mathrm{set}_i(y_i) \wedge z_i \in_i y_i \wedge z = [z_i] \wedge y = [y_i]) \\
u \leq_\sim o &\iff (\exists i \mid I(i))(\exists u_i, o_i)(\mathrm{ord}_i(u_i) \wedge \mathrm{ord}_i(o_i) \wedge u_i \leq_i o_i \wedge o = [o_i] \wedge u = [u_i]) \\
@_\sim(z, u) &\iff (\exists i \mid I(i))(\exists z_i, u_i)(\mathrm{set}_i(z_i) \wedge \mathrm{ord}_i(u_i) \wedge @_i(z_i, u_i) \wedge z = [z_i] \wedge u = [u_i])
\end{aligned}$$

Now we can show that the resulting $V_\sim$ satisfies the definition of an initial segment (see lemma C.7). And we can check that each V_i is isomorphic to some $\widehat{V_i} = (\widehat{\in}_i, \widehat{<}_i, \widehat{@}_i)$ (the structure created by the image of V_i under the map sending x to $[x]$), such that $\widehat{V_i} \leq V_\sim$ (see lemma C.8). By replacing (see lemma B.6) $V_\sim$ with an isomorphic copy extending V_0 we have effectively rendered the initial segments V_i compatible and taken their union. The proof can now be completed as indicated at the start of this section.

Appendix A. Lemmas about Well-Orderings

Definition A.1. Given relations R and R' a function f is an isomorphism of R with R' just if f is an injective surjection of $\mathrm{Ext}(R)$ with $\mathrm{Ext}(R')$ and $R(x_1, \ldots, x_n) \iff R'(f(x_1), \ldots, f(x_n))$. In particular, f is an isomorphism of the well-orders $<$ and $<'$ iff it is an isomorphism of $<$ and $<'$ as relations.

As noted before, this definition and the definition of a well-ordering requires that all well-orders have height at least 2. But, as we will have no need to use well-orders of height 0 or 1, this will cause no difficulties.

Definition A.2. Given binary relations $<$ and $<'$ say that $<'$ extends $<$ (denoted $< \subseteq_{\text{end}} <'$)[32] just if for all x, y

- $x < y \implies x <' y$
- $x <' y \wedge \text{Ext}(<)(y) \implies x < y$

$<'$ is a *proper extension of* $<$ if $<$ is an initial segment of $<'$ and $\text{Ext}(<') \not\subseteq \text{Ext}(<)$.

Note that if V and V' are initial segments then $V \leq V'$ just if $< \subseteq_{\text{end}} <'$, $\in \subseteq_{\text{end}} \in'$ and @ agrees with @$'$ on V

Lemma A.3 (Well Ordering Comparability)**.** *If $<$ and $<'$ are well-orders then there is a well-order $\widehat{<}$ such that either*

- *$<$ is isomorphic to $\widehat{<}$ and $\widehat{<}$ is an initial segment of $<'$.*
- *$<'$ is isomorphic to $\widehat{<}$ and $\widehat{<}$ is an initial segment of $<$.*

Proof. See Jech [5] pg 18-19, replacing references to sets with second order variables as appropriate. □

Next we can show that if f and h are isomorphisms between well orderings $<$ and $<'$ they must agree with each other everywhere, and that (indeed) if f and h are isomorphisms between initial segments of the structures picked out $<$ and $<'$, f and h must agree with one another on all values where both are defined.

Lemma A.4 (Well Ordering Isomorphism Agreement)**.** *Suppose $<, <', <_f, <_h, <'_f, <'_h,$ are all well-orders, with $<_f, <_h \subseteq_{end} <$ and $<'_f, <'_h \subseteq_{end} <'$, and that f is an isomorphism of $<_f$ with $<'_f$ and h is an isomorphism of $<_h$ with $<'_h$, then*

$$(\forall o \mid \text{Ext}(<_f)(o) \wedge \text{Ext}(<_h)(o))\, (f(o) = h(o))$$

Proof. Suppose, for contradiction, the lemma is false. Then, by comprehension and the definition of a well-ordering let o be the $<$-least counterexample to the claim that for all u in $Ext(<)$, if $\text{Ext}(<_f)(u) \wedge \text{Ext}(<_h)(u)$ then $f(u) = h(u)$.

Thus, we have an o such that $\text{Ext}(<_f)(o) \wedge \text{Ext}(<_h)(o) \wedge f(o) \neq h(o)$. By the fact that $<'$ is a linear order, we must have either $f(o) <' h(o)$ and $h(o) <' f(o)$.

[32]This notation reflects the fact that this definition corresponds to the notion of end-extension in model theory.

Suppose that $f(o) <' h(o)$. As $\mathrm{Ext}(<'_h))(h(o))$ and $<'_h \subseteq_{\text{end}} <'$ by the definition of extends $f(o) <' h(o)$ implies $\mathrm{Ext}(<'_h)(f(o))$ holds. As h is a surjection to $\mathrm{Ext}(<'_h)$, there must be some u such that $h(u) = f(o)$. By the fact that h is an isomorphism and $f(o) = h(u) <'_h h(o)$ we have $u <_h o$ and thus $u < o$. But by assumption we have $\mathrm{Ext}(<_f)(o) \wedge \mathrm{Ext}(<_h)(o)$. Thus, as $<_f, <_h \subseteq_{\text{end}} <$ by the definition of extends we can conclude $\mathrm{Ext}(<_f)(u) \wedge \mathrm{Ext}(<_h)(u)$. By the minimality of o $f(u) = h(u) = f(o)$ contradicting the injectivity of f

By the same argument with f and h swapped we can't have $h(o) <' f(o)$. Thus, by the definition of a well-order $h(o) = f(o)$ contradicting the assumption. □

Lemma A.5. *If $V \leq V'$ then $< \subseteq_{end} <'$*

Proof. This is immediate from part 4 of the definition of $\leq$. □

Appendix B. Lemmas About Initial Segments

Lemma 4.15 (Proper Extension Lemma)**.**

$$\Box(\forall V)\left[\mathscr{V}(V) \implies \Diamond_V(\exists V')(V' > V)\right]$$

Proof. Suppose $V = (\in, <, @)$ is an initial segment. By the layer of classes principle it is logically possible to supplement the relations $\in, <, @$ with relations $C, \in^*$ forming a layer of classes over $\mathrm{Ext}(\in, <, @)$. By applying comprehension, we can infer the possibility of a $V' = (\in', <', @')$. Informally, we construct V' by adjoining to the sets in V all classes in C not already extensionally equal to sets in V and adjoining the class of all ordinals in V to the (end of the) ordinals in V. More formally, the relations $\in', <', @'$ are given by comprehension applied to the following definitions.

$$\begin{aligned} x \in' y &\iff x \in y \vee \big[C(y) \wedge (\forall z \mid z \in^* y)(\mathrm{set}(z)) \\ &\quad \wedge (\forall z \mid \mathrm{set}(z))(\exists w \mid \mathrm{set}(w))\neg(w \in z \leftrightarrow w \in^* y) \wedge x \in^* y\big] \\ o <' o' &\iff o < o' \vee \big[C(o') \wedge (\forall u)(u \in^* o' \leftrightarrow \mathrm{ord}(u)) \wedge \mathrm{ord}(o).\big] \\ @'(x, o) &\iff @(x, o) \vee \big[\mathrm{ord}'(o) \wedge \neg\,\mathrm{ord}(o) \wedge \mathrm{set}'(x)\big] \end{aligned}$$

It is straightforward to check that these relations satisfy the definition of an initial segment, given the fact that C is disjoint from $\mathrm{Ext}(\in, <, @)$ and unpacking the definition of extends for initial segments gives us that $V' > V$. □

It is worth noting that the above reasoning allows us to establish that potentialist truth is not trivial.

Corollary B.1.

$$\Diamond \exists V \mathscr{V}(V)$$

Proof. This follows by the same construction used in the above proof applied twice to the empty structure. □

Definition B.2. Say that f is an isomorphism of $V = (\in, <, @)$ with $V' = (\in', <', @')$ just if

- f is an isomorphism of $\in$ with $\in'$
- f is an isomorphism of $<$ with $<'$
- $(\forall x \mid \mathrm{set}(x))\,(\forall o \mid \mathrm{ord}(o))\,[@(x, o) \iff @'(f(x), f(o))]$

Note that if V and V' are isomorphic, then V is an initial segment iff V' is an initial segment. Moreover, If f is an isomorphism of V with W and $V' \leq V$ then f is an isomorphism of V' with $W' \leq W$ where W' is the image of V' under f.

Lemma 4.9 (Isomorphism Agreement)**.** *Given initial segments V, $\widehat{V}$ if h, k are both isomorphisms from V to $\widehat{V}$ then $h = k$, i.e, $(\forall x \mid V(x))(h(x) = k(x))$.*

Furthermore, suppose $V_f \leq V, V_g \leq V$ and $V'_f \leq V', V'_g \leq V'$. If f is an isomorphism of V_f with V'_f and g and isomorphism of V_g with V'_g then $f(x)$ is equal to $g(x)$ on any x in both V_f and V_g

Proof. We first note that it is enough to prove the second part of the lemma, as the first part of the lemma follows from it by letting $V_f = V_g = V$, $V'_f = V'_g = V' = \widehat{V}$, $h = f$ and $g = k$.

Suppose $V, V', V_f, V_g, V'_g, V'_f, f$ and g are as in the second part of the lemma. By the Well Ordering Isomorphism Agreement lemma and lemma A.5 we know that $f(o) = g(o)$ whenever both $\mathrm{ord}_f(o)$ and $\mathrm{ord}_g(o)$. Now, for contradiction, let o be the $<$ least element such that for some set x in both V_f and V_g we have $@(x, o)$ but not $f(x) = g(x)$.

Suppose $c \in' f(x)$. As $\mathrm{set}'_f(f(x))$ by the definition of initial segment extension, it follows that $c \in'_f f(x)$. As f is an isomorphism of V_f with V'_f we have $f^{-1}(c) \in_f x$. As $V_f \leq V$ we know that $f^{-1}(c) \in x$. By hypothesis x is in V_g, so by the definition of initial segment extension again we have $f^{-1}(c) \in_g x$ and $\mathrm{set}_g(x)$.

As $@(x, o)$ and $f^{-1}(c) \in x$ by part 5 of the definition of an initial segment there is some $o' < o$ such that $@(f^{-1}(c), o')$. As x is a set in both V_f and V_g and and $f^{-1}(c) \in x$ so too is $f^{-1}(c)$. Thus, by the minimality of

o, $g(f^{-1}(c)) = f(f^{-1}(c)) = c$ and as g is an isomorphism $c \in'_g g(x)$. As $V_g \leq V'$ it follows that $c \in' g(x)$.

By the same argument if $c \in' g(x)$ then $c \in' f(x)$. Thus $f(x)$ and $g(x)$ have exactly the same elements under $\in'$. By extensionality it follows that $f(x) = g(x)$. Contradiction. □

Lemma B.3. *If V is an initial segment and $<' \subseteq_{end} <$ and V' is the restriction of V to $<'$ then $V' \leq V$ where the restriction V' to $<'$ is defined by $V' = (\in', <', @)$*[33] *and*

$$x \in' y \iff x \in y \wedge \left(\exists o \mid \text{Ext}(<')(o)\right) \left(\exists u \mid \text{Ext}(<')(u)\right) (@(x, o) \wedge @(y, u))$$

Proof. We first verify that V' is an initial segment. Parts 1, 3 of the definition are immediate from the definition of V'. The remaining parts of the definition of an initial segment follow directly from the fact that V is an initial segment.

The fact that $V' \leq V$ is immediate from the definition of extends for initial segments. □

Lemma B.4. *If V, V' are initial segments and the ordinals in V are isomorphic to the ordinals in V', then V and V' are isomorphic*

Proof. Let h be an isomorphism of $<$ with $<'$, $<_o$ be the restriction of $<$ to those elements u with $u \leq o$, and $<'_o$ be the restriction of $<'$ to elements u' with $u' \leq f(o)$. If $\text{ord}(u)$ and u is not the least ordinal in V (this extra restriction is necessary as the definition of a well-order mandates at least 2 members) define $V_u = (\in_u, <_u, @), V'_u = (\in'_u, <'_u, @)$ as in lemma B.3 (so $V_u \leq V$ and $V'_u \leq V'$). Let o be the least ordinal in V such that and V_o is defined (i.e. $(\exists u)(u < o)$) but not isomorphic to $V'_o \leq V'$. Such an ordinal must exist by comprehension and the definition of well-ordering.

Consider the case where there is at least one ordinal $u < o$ with V_u defined and isomorphic to some V'_u. We now construct an isomorphism f_o contradicting the assumption of the minimality of o.

First we define $\widehat{f}$. For x a set such that $@(x, u)$ for some $u < o$, we define $\widehat{f}(x)$ to be $f_u(x)$ if there is some $u < o$ and f_u is an isomorphism of V_u with $V'_u \leq V'$. By Isomorphism Agreement and the definition of V_u, $\widehat{f}(x)$ is well defined[34] on all such x.

Now let f_0 be equal to h on the ordinals of V_o and equal to $\widehat{f}(x)$ on any set x for which $\widehat{f}(x)$ is defined. Otherwise, if $\text{set}_o(x)$ define $f_o(x) = f_o[x]$ where

[33] Note that definition 4.6 only restricts how @ behaves on elements in V so we need not redefine @ so it is only defined on elements in V'.

[34] That means any isomorphism $f_u(x)$ of V_u with $V'_u \leq V'$ yields the same value for $\widehat{f}(x)$ allowing us to define $\widehat{f}$ via comprehension.

this denotes the pointwise image of x, i.e.,

$$f_o(x) = y \iff (\forall z)\left[z \in' y \iff (\exists w)\left(w \in x \wedge z = \widehat{f}(w)\right)\right]$$

By fatness such a y must exist and by extensionality it is unique ensuring that $f_o(x)$ is well-defined for every set in V_o.

Clearly, $\widehat{f}$ is injective and by extensionality f is easily seen to be injective. If $@'(y, u)$ for some $u < o$ then y is in the image of $\widehat{f}$ so by thinness we can infer that the image of V_o under f_o is V'_o. Clearly, $x \in_o y \iff f_o(x) \in'_o f_o(y)$ so f_o is an isomorphism contradicting the minimality of o.

We now must establish the base case: if o is the unique ordinal of height 2 in V then V_o is isomorphic to V'_o. However, by thinness and extensionality it is straightforward to verify that V_o and V'_o contain exactly two sets, the empty set and the set containing the empty set. Moreover, the function mapping the two elements that play this role in V_o to the corresponding elements in V'_o is an isomorphism. Note, this is equivalent to repeating our above construction of f_o from $\widehat{f}$ twice and taking $\widehat{f}$ to be the empty function.

This is enough to establish that V is isomorphic to V' provided there is a maximal ordinal in V. If V does not have a maximal ordinal then we can define an isomorphism f from V to V' in much the same way we defined $\widehat{f}$. That is f is equal to h on the ordinals and if $\text{set}(x)$ then (where $V_o \cong_{f_o} V'_o$ denotes the claim that f_o is an isomorphism of V_o with V'_o)

$$f(x) = y \iff (\exists o \mid \text{ord}(o))(\exists f_o)\left(f_o(x) = y \wedge V_o \cong_{f_o} V'_o\right)$$

□

Lemma 4.10 (Comparibility)**.** *Given initial segments V, V' either V is isomorphic to an initial segment $\widehat{V} \leq V'$ or V' is isomorphic to an initial segment $\widehat{V} \leq V$.*

Proof. This follows directly from Well Ordering Comparability and lemma B.4. □

Lemma B.5. *Suppose $V, V' \leq V^*$ then either $V \leq V'$ or $V' \leq V$.*

Proof. By Comparibility let h witness that either V is isomorphic to an initial segment $W \leq V'$ or V' is isomorphic to $W \leq V$. By Isomorphism Agreement applied to h and the identity function, h is the identity. □

Lemma B.6 (Isomorphism Extension)**.** *Suppose $V \leq V'$ and V is isomorphic with $\widehat{V}$, then it is logically possible that there exists a $\widehat{V}' \geq \widehat{V}$ isomorphic with V'.*

Proof. Let h be the isomorphism of V with $\widehat{V}$. We build $\widehat{V}'$ and an isomorphism f of V' with $\widehat{V}'$. If we could assume that any elements in both V' and

$\widehat{V}$ were in V we could simply define f to be equal to h on V and the identity elsewhere (letting let $\widehat{V}'$ copy V' above $\widehat{V}$). However, as we can't make this assumption we must manufacture new elements to use to supplement $\widehat{V}$.

To deal with this problem we use Superset Extendability to introduce a Layer Of Classes over $\text{Ext}(\in', <', \widehat{\in}, \widehat{<})$. These classes are guaranteed to be disjoint from every element in V' and $\widehat{V}$. We now define f to be equal to h on V and $f(x)$ equal to the singleton class of x everywhere else. If we define $\in', <', @'$ to be the isomorphic images of $\in, <, @$ under f it is straightforward to verify that $\widehat{V}' \geq \widehat{V}$ and that f is an isomorphism.

□

B.1. **Potentialistic Truth.**

Lemma 4.11 (Stability Lemma). *If $V' \geq V$ and all parameters in ϕ are sets in V then*

$$V \models_p \phi \iff V' \models_p \phi$$

.

Proof. We prove this by induction on complexity of sentences (with parameters). Suppose V, V' and ϕ are as in the lemma. The claim is evident in the case that ϕ is quantifier free.

First we prove the $\implies$ direction. So, suppose $V \models_p \phi$ and consider the following cases[35].

- **$\phi = \psi \wedge \varphi$:** By definition of $\models_p$ we have $V \models_p \psi$ and $V \models_p \varphi$. By the inductive hypothesis $V' \models_p \psi$ and $V' \models_p \varphi$. By the definition of $\models_p$ again $V' \models_p \psi \wedge \varphi$.
- **$\phi = \neg\psi$:** This follows by the same reasoning as in the above case.
- **$\phi = \exists x \psi(x)$:** By definition of $\models_p$ it is logically possible that there is some $\widehat{V} \geq V$ and a set a in $\widehat{V}$ such that $\widehat{V} \models_p \psi(a)$. By Comparibility there is some V^* such that either $V^* \leq V'$ and V^* is isomorphic to $\widehat{V}$ or $V^* \leq \widehat{V}$ and V^* is isomorphic to V'.

  In the first case let f witness V^* is isomorphic to $\widehat{V}$. As $\models_p$ is preserved under isomorphism, we have $V^* \models_p \psi(f(a), f(a_1), \ldots, f(a_n))$ where $a_1, \ldots, a_n$ are the parameters appearing in ψ. By Isomorphism Agreement $V \leq V^*$ and f is the identity on V, thus $V^* \models_p \psi(f(a))$. By the inductive hypothesis $V' \models_p \psi(f(a))$, hence it is logically possible there is some $V'' \geq V'$ such that $V'' \models_p \psi(f(a))$.

[35] Note that it is sufficient to consider these cases as we view all other logical connectives/quantifiers as abbreviations defined in terms of the above connectives/quantifiers.

In the second case, by Isomorphism Extension, it is logically possible there is some $V'' \geq V'$ and an isomorphism f of $\widehat{V}$ with V''. By the same argument above we can infer that $V'' \models_p \psi(f(a))$.

Note that S5 allows us to go from $\Diamond\Diamond\rho$ to $\Diamond\rho$ so we can infer it is simply logically possible that $V'' \models_p \psi(f(a))$. By the definition of $\models_p$ the logical possibility of $V'' \geq V'$ containing a set $f(a)$ such that $V'' \models_p \psi(f(a))$ entails that $V' \models_p \exists x\psi(x)$.

Note that the above cases are exhaustive as all other logical operations are defined as abbreviations expressed in terms of the above operations.

For the $\Longleftarrow$ direction, suppose that $V' \models_p \phi$. If $\phi = \psi \wedge \varphi$ or $\phi = \neg\psi$ the proof is just as above. If $V' \models_p \exists x\psi(x)$ by the definition of $\models_p$ it is logically possible there is some $V'' \geq V'$ and set a in V'' such that $V'' \models_p \psi(a)$. As $V'' \geq V$ it follows immediately that $V \models_p \exists x\psi(x)$

□

Lemma 4.12 (Bounded Quantifiers Lemma). *If V is an initial segment then*

$$[V \models_p (\forall x)(x \in a \to \phi(x,a))] \iff [(\forall x)(x \in a \to V \models_p \phi(x,a))]$$

Proof. ($\Longrightarrow$) Suppose $V \models_p (\forall x)(x \in a \to \phi(x,a))$. Unpacking definitions gives that

$$\Box_V (\forall \widehat{V})(\forall x)\left[\widehat{V} \geq V \wedge \widehat{\text{set}}(x) \wedge x \widehat{\in} a \to \widehat{V} \models_p \phi(x,a)\right]$$

By the fact that what's necessary is actual by instantiating $\widehat{V}$ with V we have $(\forall x)\,[\text{set}(x) \wedge x \in a \to V \models_p \phi(x,a)]$. Since, by definition whenever $x \in a$ we have set(x), this yields the desired conclusion.

($\Longleftarrow$) Suppose $(\forall x)\,(x \in a \to V \models_p \phi(x,a))$. Now consider an arbitrary logically possible scenario in which we have some $\widehat{V} \geq V$ and set x in $\widehat{V}$ such that $x \widehat{\in} a$. By the definition of extends and the fact that set(a), we can infer $x \in a$. Thus, $V \models_p \phi(x,a)$.

By Stability Lemma this implies $\widehat{V} \models_p \phi(x,a)$. As $\widehat{V}$ and x were arbitrary satisfying $\widehat{V} \geq V$ and $x \widehat{\in} a$ we may conclude.

$$\Box_V (\forall \widehat{V})(\forall x)\left[\widehat{V} \geq V \wedge \widehat{\text{set}}(x) \wedge x \widehat{\in} a \to \widehat{V} \models_p \phi(x,a)\right]$$

That is, $V \models_p (\forall x)(x \in a \to \phi(x,a))$. □

Appendix C. $V_\sim$ is an initial segment

Lemma C.1. *The relation $\sim$ is an equivalence relation on the objects in the V_i's, i.e. on those x satisfying $(\exists i \mid I(i))(V_i(x))$.*

Proof. It is easily established that $\sim$ is reflexive. Given x such that for some i, $V_i(x)$ then, by the definition of $\sim$, the identity function on V_i witnesses $x \sim x$.

To show symmetry note that if $x \sim y$ is witnessed by $V_i^* \leq V_i, V_j^* \leq V_j$ and isomorphism f of V_i^* with V_j^* with $f(x) = y$ then $y \sim x$ is witnessed by $V_j^* \leq V_j, V_i^* \leq V_i$ and f^{-1}.

To show that $\sim$ is transitive suppose $x \sim y$ and $y \sim z$. If one of x, y or z is in V_0, it is straightforward to infer (using Isomorphism Agreement and the fact that $V_0 \leq V_i$ for all i in I) that all of x, y and z are in V_0 and $x = y = z$ so $x \sim z$. So suppose none of x, y or z are in V_0 and $V_i(x), V_j(y), V_k(z)$.

By the definition of $\sim$ and the fact that V_i, V_j, V_k are disjoint over V_0, let f be an isomorphism from $V_i' \leq V_i$ to $V_j' \leq V_j$ with $f(x) = y$ witnessing $x \sim y$. Similarly, let g be an isomorphism from $V_j^* \leq V_j$ to $V_k^* \leq V_k$ with $g(y) = z$.

By lemma B.5 either $V_j' \leq V_j^*$ or $V_j^* \leq V_j'$. In the first case $g \circ f$ is an isomorphism of V_i' with some $V_k' \leq V_k$ with $g \circ f(x) = z$ so $x \sim z$. In the later case $f^{-1} \circ g^{-1}$ is an isomorphism of V_k^* with some $V_i^* \leq V_i$ and $f^{-1} \circ g^{-1}(z) = x$ so $z \sim x$ and by symmetry $x \sim z$ □

Definition C.2 ($\widehat{V_i}$)**.** Define $\widehat{V_i} = (\widehat{\in}_i, \widehat{<}_i, \widehat{@}_i)$ to be the structure given by restricting the relations $\in_\sim, <_\sim, @_\sim$ to the image of V_i under $[\cdot]$. Formally, we define $\widehat{\in}_i, \widehat{<}_i, \widehat{@}_i$ via comprehension to satisfy:

$$\begin{aligned} a \widehat{\in}_i b &\iff (\exists a_i, b_i)\,(\mathrm{set}_i(a_i) \wedge \mathrm{set}_i(b_i) \wedge a = [a_i] \wedge b = [b_i] \wedge a \in_\sim b) \\ o \widehat{<}_i u &\iff (\exists o_i, u_i)\,(\mathrm{ord}_i(o_i) \wedge \mathrm{ord}_i(u_i) \wedge o = [o_i] \wedge u = [u_i] \wedge o <_\sim u) \\ \widehat{@}_i(a, o) &\iff (\exists a_i, o_i)\,(\mathrm{set}_i(a_i) \wedge \mathrm{ord}_i(o_i) \wedge a = [a_i] \wedge o = [o_i] \wedge @_\sim(a, o)) \end{aligned}$$

Lemma C.3. *If i in I the function $[\cdot]$ taking x to $[x]$ is an isomorphism of V_i with $\widehat{V_i}$.*

Proof. The definition of $\widehat{V_i}$ defines $\widehat{V_i}$ as the image of V_i under $[\cdot]$ so $[\cdot]$ is clearly surjective.

To show injectivity suppose $[a] = [b]$ for where $V_i(a), V_i(b)$. Thus $a \sim b$ and by the definition of $\sim$ there is an isomorphism f from $V_j^* \leq V_j$ to $V_k^* \leq V_k$ for some j, k in I with $f(a) = b$.

We consider two cases. First, suppose $V_0(a)$ or $V_0(b)$ and $[a] = [b]$. If $V_0(a)$ then, by Isomorphism Agreement as $V_0, V_j^* \leq V_j$ and $V_0, V_k^* \leq V_k$, f agrees with the identity map on V_0 wherever they are both defined. Hence $a = b$. If $V_0(b)$ the same argument applied to f^{-1} gives us $a = b$

Now, suppose, neither a or b is in V_0. By hypothesis $V_i(a)$ and $V_j(a)$ and as a isn't in V_0 disjointness over V_0 implies $j = i$. Similarly, $V_k(b)$ and $V_i(b)$ implies $k = i$ so $j = k = i$ By Isomorphism Agreement f must be the identity on V_i giving $a = b$. This establishes injectivity.

We now show the $\implies$ direction of the definition of isomorphism for relations. Clearly if $x_i \in_i y_i$ then $[x_i] \in_\sim [y_i]$ and thus $[x_i]\widehat{\in}_i[y_i]$. A similar argument establishes the $\implies$ direction for $<_i$ and $@_i$.

For the $\impliedby$ direction of the definition of isomorphism suppose $[x_i]\widehat{\in}_i[y_i]$. In this case, $[x_i] \in_\sim [y_i]$ and by definition of $V_\sim$ there is some j in I and sets x_j, y_j in V_j with $x_j \in_j y_j$ and $[x_j] = [x_i] \wedge [y_j] = [y_i]$. Thus, $y_j \sim y_i$ and by the definition of $\sim$ there is some $V_j^* \leq V_j$ and $V_i^* \leq V_i$ and isomorphism f of V_j^* with V_i^* with $f(y_j) = y_i$.

By the definition of extends for initial segments as $x_j \in_j y_j$ and $\mathrm{set}_j^*(y_j)$ we have $x_j \in_j^* y_j$. As f is an isomorphism and $V_i^* \leq V_i$ we have $f(x_j) \in_i f(y_j) = y_i$. But, by definition of $\sim$ (definition 6.2) $f(x_j) \sim x_j \sim x_i$ and by transitivity $[f(x_j)] = [x_i]$. Hence, as $[\cdot]$ is injective $x_i = f(x_j) \in_i f(y_j) = y_i$.

A similar argument establishes the $\impliedby$ direction of the definition of isomorphism for $\widehat{<}_i$ and $\widehat{@}_i$ completing the proof. □

Note that this also establishes that $\hat{V}_i$ is an initial segment.

Lemma C.4. *Given i, j in I either $\widehat{V}_i \leq \widehat{V}_j$ or $\widehat{V}_j \leq \widehat{V}_i$.*

Proof. By Comparibility we can assume, without loss of generality, that there is some initial segment $V \leq V_j$ and isomorphism f from V_i to V. By definition of $\sim$ we have that for any set x in V_i, $[x] = [f(x)]$ ensuring that $\mathrm{set}_i \subseteq \mathrm{set}_j$ and $\mathrm{ord}_i \subseteq \mathrm{ord}_j$. As f respects $\in_i, <_i, @_i$ and $V \leq V_j$ we can infer that $\widehat{V}_i \leq \widehat{V}_j$. □

Lemma C.5. *Given $\widehat{V}_i = (\widehat{\in}_i, \widehat{<}_i, \widehat{@}_i)$ (as in definition C.2) then*

$$\begin{aligned} \widehat{\mathrm{set}}_i(y) &\implies \big(x \in_\sim y \iff x\widehat{\in}_i y\big) \\ \widehat{\mathrm{ord}}_i(u) &\implies \big(o <_\sim u \iff o\widehat{<}_i u\big) \\ \widehat{\mathrm{ord}}_i(u) &\implies \Big(@_\sim(y,u) \iff \widehat{@}_i(y,u)\Big) \end{aligned}$$

Proof. We prove the first claim and note the other two claims follow by similar reasoning. So consider an arbitrary i in I and y such that $set_i(y)$.

($\impliedby$) This is is immediate from the definition of $\widehat{V}_i$.

($\implies$) Suppose $x \in_\sim y$. By the definition of $\in_\sim$ there is some j in I and sets x_j, y_j in V_j such that $x = [x_j]$, $y = [y_j]$ and $x_j \in_j y_j$. Hence $[x_j] = x \mathbin{\widehat{\in}_j} y = [y_j]$.

By lemma C.4 either $\widehat{V}_j \leq \widehat{V}_i$ or $\widehat{V}_i \leq \widehat{V}_j$. In the former case by the definition of extends we have $x \widehat{\in}_i y$. In the later case, as $\widehat{\mathrm{set}_i}(y)$ the definition of extends gives us $x \widehat{\in}_j y$. ☐

Lemma C.6.

$$\mathrm{set}_\sim(x) \iff (\exists i \mid I(i)) \left(\widehat{\mathrm{set}_i}(x)\right)$$
$$\mathrm{ord}_\sim(o) \iff (\exists i \mid I(i)) \left(\widehat{\mathrm{ord}_i}(o)\right)$$

Proof. This is immediate from the definition of $V_\sim$ and the definition of $\widehat{V}_i$. ☐

We now formalize the reasoning in section 6 establishing that $V_\sim$ is an initial segment.

Lemma C.7. *$V_\sim$ is an initial segment.*

Proof. Recalling the definition of initial segment (definition 4.6) gives us 7 clauses to check.

(1) Suppose X is a non-empty collection of objects satisfying $\mathrm{ord}_\sim$. Let u witness that X is non-empty. By lemma C.3 for some i in I $\widehat{\mathrm{ord}_i}(u)$. Furthermore, if $o <_\sim u$ then $o \widehat{<}_i u$. As $\widehat{<_i}$ is a well-order it follows that X has a $\widehat{<}_i$ least element which is therefore a $<_\sim$ least element.

If $\mathrm{ord}_\sim(o)$ and $\mathrm{ord}_\sim(u)$ then, by lemma C.6, let i, j in I such that $\widehat{\mathrm{ord}_i}(o)$ and $\widehat{\mathrm{ord}_j}(u)$. Without loss of generality by lemma C.4 $\widehat{V}_i \leq \widehat{V}_j$ and as $\widehat{<}_j$ is a linear order we must have o, y comparable with respect to $\widehat{<}_j$. Their comparability with respect to $<_\sim$ follows from lemma C.5 that o and u are comparable.

To show transitivity, suppose we have $a <_\sim b$ and $b <_\sim c$. By lemmas C.5 and C.6 there are i, j with $a \widehat{<}_i b$ and $b \widehat{<}_j c$. By lemma C.4 without loss of generality $V_i \leq V_j$ and by the definition of extends and transitivity in V_j it follows that $a \widehat{<}_j c$ and thus by lemma C.3 $a <_\sim c$. Irreflexivity follows by a similar argument.

(4) Suppose $ord_\sim(o)$ and for all y with $X(y)$ there is some $o' <_\sim o$ such that $@_\sim(y, o')$. By lemma C.6 let i in I be such that $o = [o_i]$ for o_i an ordinal in V_i.

By lemma C.5 if $o' <_\sim o$ and $@_\sim(y, o')$ then $o' \widehat{<}_i o$ and $\widehat{@}_i(y, o')$. Thus by fatness applied in $\widehat{V}_i$ there is some x with $\widehat{@}_i(x, o)$ containing (under $\widehat{\in}_i$) just those y in X. By lemma C.5 x also has exactly the objects satisfying X for members in $V_\sim$ and $@_\sim(x, o)$.

(2, 3, 5, 6) These follow by a applying the respective properties in the initial segments $\widehat{V}_i$ using lemmas C.3, C.6, C.4 and the definition of $V_\sim$ by similar reasoning as above.

□

Lemma C.8. *If $I(i)$ then $\widehat{V}_i \leq V_\sim$*

Proof. This is immediate from lemmas C.7 C.5 and the definition of extends. □

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