Chapter 12

Defense of the ZFC Axioms

Finally, it remains to show that my potentialist translations of the ZFC axioms of set theory can be proved using my inference rules for logical possibility.

I will frequently use iterated applications of the $\Box$ and $\Diamond$ Collapsing Lemmas (proved in sections 8.1 and 8.3) to simplify the translation of set theoretic sentences. Recall that the $\Box$ Collapsing Lemma says:

"If ϕ_2 and θ are content restricted to $\mathcal{L}_1, \mathcal{L}_2$ and ϕ_1 is content restricted to $\mathcal{L}_0, \mathcal{L}_1$, then we have

$\vdash \Box_{\mathcal{L}_0}(\phi_1 \rightarrow \Box_{\mathcal{L}_1}(\phi_2 \rightarrow \theta)) \leftrightarrow \Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \rightarrow \theta)$"

This lets us simplify the translation of set theoretic statements with repeated $\forall$ quantifiers by replacing a string of $\Box$ statements with a single $\Box$ statement (and similarly with $\Diamond$ statements.[1].

So, for instance, a set theoretic claim of the form $(\forall x)(\forall y)(\phi)$ gets

[1]Translations for strings of repeated $\exists$ quantifiers which becomes strings of $\Diamond$ statements are collapsed into a single $\Diamond$ using the $\Diamond$ collapsing lemma similarly

translated as follows,

$$\Box(\mathscr{V}(\vec{V_0}) \to \Box_{\vec{V_0}}[\vec{V_1} \geq_{\mathbf{x}} \vec{V_0} \to \Box_{\vec{V_1}}(\vec{V_2} \geq_{\mathbf{y}} \vec{V_1} \to t_2(\phi))])$$

However, it is provably equivalent to the following simpler sentence, via two applications of the $\Box$ Simplification Lemma[2]. (The fact that that the sentence inside each $\Box_{V_i}$ or $\Diamond_{V_i}$ subformula in the translation of a set theoretic sentence ϕ is always content-restricted to V_i, V_{i+1} ensures that the premises of the above Lemma are satisfied).

$$\Box(\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\mathbf{x}} \vec{V_0} \wedge \vec{V_2} \geq_{\mathbf{y}} \vec{V_1} \to t_2(\phi)])$$

In what follows, I will sketch the reasoning used to prove relevant propositions, but leave it to the reader to fill in the technical details such as applying the wrapping trick or subscripting relations to mimic quantifying in.

[note that by my abbreviations in $f(y) = y$, ONLY the right hand token is a genuine variable]

12.1 Foundation and Other Easy Cases

Proposition 12.1.1. ***Foundation*** $(\forall x)[(\exists a)(a \in x) \to (\exists y)(y \in x \wedge \neg(\exists z)(z \in y \wedge z \in x))]$ *Translating this and then simplifying with $\Diamond$-Collapsing Lemma as above yields:* $\Box[\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\boldsymbol{x}} \vec{V_0} \wedge \Diamond_{\vec{V_1}}[\vec{V_2} \geq_{\boldsymbol{a}} \vec{V_1} \wedge f_2(a) \in f_2(x)] \to \Diamond_{\vec{V_1}}[\vec{V_2} \geq_{\boldsymbol{y}} \vec{V_1} \wedge f_2(y) \in f_2(x) \wedge \neg \Diamond_{\vec{V_2}}(\vec{V_3} \geq_{\boldsymbol{z}} \vec{V_2} \wedge f(z) \in_3 f_3(y) \wedge f_3(z) \in_3 f_3(x))])]$

This essentially says: if V_1, f_1 can be extended such that $f_1(a)$ is $\in_2 f_2(x)$, then it could alternatively be extended by a V_2, f_2 whose assignment for y ensures that no further extension V_3, f_3 can assign f_3 of z such that

[2] The trick is to first use the $\Box$ Simplifying Lemma to simplify the innermost statement, in this case,$\Box_{V_0}[V_1 \geq_{\mathbf{x}} V_0 \to \Box_{V_1}(V_2 \geq_{\mathbf{y}} V_1 \to \phi)]$, and then to proceed progressively outward [SAY MORE?]

$f_3(z) \in_3 f_3(y) \wedge f_3(z) \in_3 f_3(x)$.

To this end, we prove the following lemma.

Lemma 12.1.2. $\mathscr{V}(V) \to (\forall x)[(\exists a)(a \in x) \to (\exists y)(y \in x \wedge \neg(\exists z)(z \in y \wedge z \in x))]$

Proof. Assume that $\mathscr{V}(V)$. Consider an arbitrary x, such that $set(x)$ and $(\exists a)(a \in x)$. By the fact that the ords are well ordered by $\leq$ (as defined in 7.1), there will be some $\leq$-least member of ord o with the following property: there exists y at level o and $y \in x$. Any $z \in y$ occurs at some level $o' < o$, by the fact that $\mathscr{V}(V)$. Thus, by minimality of o, $\neg z \in x$. Thus we have $y \in x$ such that $\neg(\exists z)(z \in y \wedge z \in x)$, as desired. □

Proof. Now we will prove the proposition using the lemma above. Consider an arbitrary situation in which $\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\mathbf{x}} \vec{V_0} \wedge \Diamond_{\vec{V_1}}[\vec{V_2} \geq_{\mathbf{a}} \vec{V_1} \wedge f_2(a) \in f_2(x)]$.

Note that if $f_1(x)$ is the empty set, then it is not possible (fixing the facts about $\vec{V_1}$) to have $\vec{V_2} \geq_a \vec{V_1}$ with $f_2(a) \in f_2(x)$. Thus, we may assume $f_1(x)$ is not the empty set. Thus, by the above lemma (and simplified choice), we can choose a y such that $y \in_1 x \wedge \neg(\exists z)(z \in_1 y \wedge z \in_1 x)$.

[can finish by just using new lemma here]

We can then let $V_2 = V_1$ and f_2 to be just like f_1, except that $f_2(y) = y$. Thus we have $f_2(y) \in f_2(x) \wedge (\forall z)\neg(z \in_2 f_2(y) \wedge z \in_2 f_2(x)$.

Now, suppose for contradiction that it were $\Diamond_{\vec{V_2}}$ to have $\vec{V_3} \geq_z \vec{V_2}$ with $f_3(z) \in_3 f_3(y) \wedge f_3(z) \in_3 f_3(x)$. Then we would have $f_3(z) \in_3 f_2(x)$ and $f_3(z) \in_2 f_2(y)$ [by the fact that $\vec{V_3} \geq_z \vec{V_2}$]. But this contradicts our choice for

$f_2(y)$, specifically, the fact that $(\forall z)\neg(z \in_2 f_2(y) \wedge z \in_2 f_2(x)$[3].

Thus we can conclude that $\Diamond_{\vec{V_1}}[\vec{V_2} \geq_{\mathbf{y}} \vec{V_1} \wedge f_2(y) \in f_2(x) \wedge \neg \Diamond_{\vec{V_2}}(\vec{V_3} \geq_{\mathbf{z}} \vec{V_2} \wedge f(z) \in_3 f_3(y) \wedge f_3(z) \in_3 f_3(x))]$, as desired.

$\square$

Potentialist versions of Extensionality, Pairing, Powerset, Union and Choice can be proved in much the same way noted above, by using the fact that the corresponding principle must hold within any initial segment V_i such that $\mathscr{V}(V_i)$.

Proposition 12.1.3 (Extensionality)**.** $(\forall x)(\forall y)[(\forall z)(z \in x \leftrightarrow z \in y) \rightarrow x = y]$ *Translating this and then simplifying via* $\square$ *collapsing (and a little FOL) yields* $\square[\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_x \vec{V_0} \wedge \vec{V_2} \geq_{\boldsymbol{y}} \vec{V_1} \wedge \square_{\vec{V_2}}(\vec{V_3} \geq_{\boldsymbol{z}} \vec{V_2} \rightarrow [f_3(z) \in_3 f_3(x) \leftrightarrow f_3(z) \in_3 f_3(y)]) \rightarrow f_2(x) = f_2(y)]$

Informally, this says that $f_2(x)$ *and* $f_2(y)$ *are assigned in such a way that any extending* $\vec{V_3} \geq_z \vec{V_2}$ *must satisfy* $f_3(z) \in f_3(x) \leftrightarrow f_3(z) \in f_3(y)$*, then* $f_2(x) = f_2(y)$*.*

Proof. I will prove this claim by exploiting the fact that extensionality holds *inside* any relevant V_2 such that $\mathscr{V}(\vec{V_2})$ (because Thinness includes an extensionality requirement) to argue that $f_2(x) = f_2(y)$.

Assume that $\vec{V_0}, \vec{V_1}, \vec{V_2}$ satisfy $\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\mathbf{x}} \vec{V_0} \wedge \vec{V_2} \geq_{\mathbf{y}} \vec{V_1}$ and $\square_{\vec{V_2}}(\vec{V_3} \geq_{\mathbf{z}} \vec{V_2} \rightarrow [f_3(z) \in_3 f_3(x) \leftrightarrow f_3(z) \in_3 f_3(y)])$.

Now suppose for contradiction that $\neg f_2(x) = f_2(y)$. By the fact that V_2 satisfies extensionality there is some $set_2(k)$ such that $\neg(k \in_2 f_2(x) \leftrightarrow k \in_2$

[3]Note that this sentence is content restricted to V_2 so it must remain true in our current context

$f_2(y))$. Thus, it is possible (holding $\vec{V_2}$ fixed) that $\vec{V_3} \geq_{\mathbf{z}} \vec{V_2}$ and $f_3(\mathbf{z})$ applies to such a *set*$_2$ k.[4] However, (by unpacking the definition of $\vec{V_3} \geq_z \vec{V_2}$) it follows that this scenario must be one in which $\neg[f_3(z) \in_3 f_3(x) \leftrightarrow f_3(z) \in_3 f_3(y)]$), contrary to the $\Box_{\vec{V_2}}$ assumption above.[5]

Thus, we have a our desired proof by contradiction that $f_2(x) = f_2(y)$. And since $\vec{V_0}, \vec{V_1}, \vec{V_2}$ are arbitrary, we can derive that the above statement holds with logical necessity.[6] □

Proposition 12.1.4 (Union)**.** *"$\forall z\, \exists a\, \forall y\, \forall x[(x \in y \wedge y \in z) \Rightarrow x \in a]$."*

Translating and then applying the □ Collapsing Lemma gives

$\Box(\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\boldsymbol{z}} \vec{V_0} \to \Diamond_{\vec{V_1}}[\vec{V_2} \geq_{\boldsymbol{a}} \vec{V_1} \wedge \Box_{\vec{V_2}}(\vec{V_3} \geq_{\boldsymbol{x}} \wedge \vec{V_2} \wedge V_4 \geq_{\boldsymbol{y}} \vec{V_3} \to [f_4(x) \in_4 f_4(y) \wedge f_4(y) \in f_4(z) \to f_4(x) \in f_4(a)]])])$.

Thus it essentially says that for any V_1, f_1 assigning z, there is an extension V_2, f_2 which assigns a to a 'union set' for $f_1(z)$.[7]

Proof. As before, we will prove the needed conclusion by exploiting the fact that Union holds true within *within* any V_1 such that $\mathscr{V}(\vec{V_1})$. Consider an

[4] We know this by Simple Choice and the Multiple Definitions Lemma.

[5] Specifically, by the Simpler Choice Lemma it is logically possible that the otherwise unused predicate $P(z)$ applies to a unique object z satisfying the formula above. Entering this $\Diamond_{\vec{V_2},P}$ context and applying simple comprehension a few times (as per the Multiple Definitions Lemma), it is logically possible that $V_3 =_{set} V_2$ and $(\forall k)(f_3(\mathbf{z}) = k \leftrightarrow P(z))$ and that $(\forall y)(\neg y = \mathbf{z}) \to f_3(y) = f_2(y))$ for all other values of y.

Enter this $\Diamond_{\vec{V_0},\vec{V_1},\vec{V_2},P}$ context. The fact that $(\forall k)[P(k) \to \neg(z \in_2 f_2(x) \leftrightarrow z \in_2 f_2(y))]$ is content-restricted to V_2 so it can be imported into this context. Combining this with our specification that $f_3(\mathbf{z})$ is the unique object satisfying $P(x)$ and $f_3 = f_2$ on all other values, we get $\neg[\vec{V_3} \geq_{\mathbf{z}} \vec{V_2} \to [f_3(\mathbf{z}) \in_3 f_3(\mathbf{x}) \leftrightarrow f_3(\mathbf{z}) \in_3 f_3(\mathbf{y})]$.

Leaving this $\Diamond_{\vec{V_0},\vec{V_1},\vec{V_2},P}$ context, Inn$\Diamond$ allows us to conclude that $\Diamond_{\vec{V_1},\vec{V_2}} \neg[\vec{V_3} \geq_{\mathbf{z}} \vec{V_2} \to [f_3(z) \in_3 f_3(x) \leftrightarrow f_3(z) \in_3 f_3(y)]$. But our $\Box_{\vec{V_1},\vec{V_2}}(\vec{V_3} \geq_{\mathbf{z}} \vec{V_2} \ldots)$ assumption above is the negation of this claim.

[6] That is, we can derive this conclusion via □I because all the 0 assumptions used to secure this result are content restricted to the empty list.

[7] in the sense that extending V_3, f_3, V_4, f_4 which assign x and then y so that $f_4(x) \in_4 f_4(y) \in_4 f_4(z)$ must also satisfy $f_4(x) \in_4 f_4(a)$)

arbitrary scenario in which $\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\mathbf{z}} \vec{V_0}$. We can derive the fact that there is unique $set_1(w)$ such that $(\forall k)[k \in_1 w \leftrightarrow (\exists k')(k \in_1 k' \wedge k' \in f_1(x)]$ from the fact that $\mathscr{V}(V_1)$ as follows. It is logically possible that, [review wording] letting H stand fo some otherwise-unused one place relation symbol, (given the facts about $\vec{V_1}$) that $(\forall k)(H(k) \leftrightarrow \exists k' k \in_1 k' \wedge k' \in f_1(x))$ by comprehension. We can deduce that $f_1(o)$ occurs at some ordinal level and everything everything that satisfies H occurs at a lower level than o. Thus, by the thickness property of $\mathscr{V}(\vec{V_1})$, we have that there is a set_1 w occuring at level o which contains exactly the elements of H. Thus we have that there is a $set_1(w)$ such that $(\forall k)(k \in_1 w \leftrightarrow (\exists k')k \in_1 k' \wedge k' \in f_1(x))$. Now the above claim is this sentence is content-restricted to V_1 it must have been true in our original scenario.

Thus, there is a$set_1(w)$ which behaves like a union set for $f_1(x)$ as above. By Simple Comprehension (and the Multiple Definition Lemma) and it is logically possible (given the facts about V_1, f_1) to have $\vec{V_2} \geq_a \vec{V_1}$ such that $V_2 =_{set} V_1$ and $f_2(a)$ is this set.[8]

It now is straightforward to verify that V_1, V_2 witness the desired relationship.[9] □

Proposition 12.1.5 (Pairing). *"$\forall x \forall y \exists z(x \in z \wedge y \in z)$" Translating and*

[8] i.e, it is the unique set_1 w such that $(\forall k)(k \in_1 w \leftrightarrow (\exists k')k \in_1 k' \wedge k' \in f_1(x))$

[9] To check that this choice for $f_2(a)$ behaves as desired, consider an arbitrary scenario (holding facts about $\vec{V_2}$ fixed) in which $\vec{V_3} \geq_{\mathbf{x}} \vec{V_2} \wedge V_4 \geq_{\mathbf{y}} \vec{V_3}$ such that $f_4(x) \in_4 f_4(y) \wedge f_4(y) \in_4 f_4(z))$. By the fact that $f_4(z) = f_3(z) = f_2(z)$ we have $f_4(y) \in_2 f_2(z)$, and thus $f_4(x) \in_2 f_4(y)$. Now, our characterization of $f_2(a)$ above is content-restricted to $\vec{V_2}$, so it must remain true in the current context. Thus we have: $(\forall k)(k \in_2 f_2(a) \leftrightarrow (\exists k')k \in_2 k' \wedge k' \in f_2(z))$. Putting these facts together we can derive $f_4(x) \in_2 f_2(a)$ and hence $f_4(x) \in_4 f_4(a)$, as desired.

The reader can now see how the result follows.

then applying the $\Box$ *Collapsing Lemma gives* $\Box[\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\boldsymbol{x}} \vec{V_0} \wedge \vec{V_2} \geq_{\boldsymbol{y}} \vec{V_1} \rightarrow \Diamond_{\vec{V_1}}(\vec{V_3} \geq_{\boldsymbol{z}} \vec{V_3} \wedge f_3(x) \in_3 f_3(z) \wedge f_3(y) \in_3 f_3(z)))$

Thus it essentially says that any V_2, f_2 *assigning* x *and* y *can be extended by a* V_3, f_3 *assigning* z *such that* $f_3(z)$ *contains exactly* $f_2(x)$ *and* $f_2(y)$

Proof. Consider an arbitrary situation in which $\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\mathbf{x}} \vec{V_0} \wedge \vec{V_2} \geq_{\mathbf{y}} \vec{V_1}$.

By the fact that $\mathscr{V}(V_2)$ and the One More Layer Lemma**??**, we can have (while holding fixed the facts about V_2) a V_3 which extends V_2 by adding one more layer of classes [10]. By simple comprehension, it would be possible (holding fixed the facts about $\vec{V_2}, \vec{V_3}$) for a predicate P to apply to exactly those objects z such that $z = f_2(x) \vee z = f_2(y)$. Thus V_3 includes a (unique) set_3 k whose sole elements are $f_2(x)$ and $f_2(y)$.[11] Now by Simple Comprehension and the Multiple Definitions Lemma, it is $\Diamond_{\vec{V_2}}$ to have f_3 such that $\vec{V_3} \geq_{\mathbf{z}} \vec{V_3}$ except for $f_3(z)$ = the unique set_3 whose elements are exactly $f_2(x)$ and $f_2(y)$.

Entering this $\Diamond_{\vec{V_2}}$ context and using first order logic to unpack definitions yields the desired conclusion that $\vec{V_3} \geq_{\mathbf{z}} \vec{V_3} \wedge f_3(x) \in_3 f_3(z) \wedge f_3(y) \in_3 f_3(z)$.

Exiting this $\Diamond_{\vec{V_2}}$ context [inc? and pulling out the above, suitably content-restricted conclusion], and completing our conditional argument yields $\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\mathbf{x}} \vec{V_0} \wedge \vec{V_2} \geq_{\mathbf{y}} \vec{V_1} \rightarrow \Diamond_{\vec{V_2}}(\vec{V_3} \geq_{\mathbf{z}} \vec{V_3} \wedge f_3(x) \in_3 f_3(z) \wedge f_3(y) \in_3 f_3(z))$. Finally, since we proved this from empty assumption, it holds with logical necessity, as above. $\square$

[10] In essence this is a scenario where we have a layer of classes over all the objects in $Ext(V_2)$ and then take set_3 apply to all the set_2s plus all of the classes which are not co-extensive with some already existing set_2, and define everything else in the obvious way

[11] [make full sntence]since there is a class with this property, and that class is either a set_3 itself or has exactly the same elements as some set_2

Proposition 12.1.6 (Powerset)**.** *"$\forall x \exists y \forall z[z \subseteq x \to z \in y]$" That is, "$\forall x \exists y \forall z[(\forall w)(w \in z \to w \in x) \to z \in y]$"*

Translating and simplifying this with $\Box$ collapsing yields: $\Box[\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\boldsymbol{x}} \vec{V_0} \to \Diamond_{\vec{V_1}}[\vec{V_2} \geq_{\boldsymbol{y}} \vec{V_1} \wedge \Box_{\vec{V_2}}(\vec{V_3} \geq_{\boldsymbol{z}} \vec{V_2} \to \Box_{\vec{V_3}}[V_4 \geq_{\boldsymbol{w}} \vec{V_3} \to (f_4(w) \in_4 f_4(z) \to f_4(w) \in_4 f_4(x))] \to f_3(z) \in_3 f_3(y)])])$.

This intuitively says that for any initial segment and assignment V_1, f_1 we can have an extending $\vec{V_2} \geq_{\boldsymbol{y}} V_1$ which assigns $f_2(y)$ to the powerset of $f_1(x)$ (where the latter notion is understood in a modal sense).[12]

Proof. Consider an arbitrary situation in which $\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\mathbf{x}} \vec{V_0}$. As before, we know by $\mathscr{V}(\vec{V_1})$ and the One More Layer of Classes Lemma**??** we can have a V_2 which contains a set_2 whose elements are exactly the set_1s such that $[(\forall a)(a \in_1 b \to b \in_1 f_1(x))]$[13]. By making this choice for $f_2(y)$, we can have: $\Diamond_{\vec{V_1}}(\vec{V_2} \geq_{\mathbf{y}} \vec{V_1} \wedge V_2$ contains a single layer of classes over $\vec{V_1} \wedge f_2(y)$ contains all subsets of $f_1(x)$ in the sense of V_1).

Entering this $\Diamond$ scenario, we can deduce that $f_2(y)$ also contains all subsets of $f_2(x)$ *in the sense of* V_2, i.e., $\Box_{\vec{V_2}}[(\forall k)(C(k) \to k \in_2 f_2(x)) \to (\exists k')(k' \in_2 f_2(y) \wedge (\forall k)[C(k) \leftrightarrow k \in_2 k'])]$, ([review wording] proving this fact will be helpful, because it is context-restricted to V_2, hence can be imported into into any context where the V_2 facts are held fixed.) .[14].

[12] Specifically, if any extending $\vec{V_3} \geq_{\mathbf{z}} \vec{V_2}$ which assigns $f_3(z)$ to something that behaves like a subset of x (in the sense that any $\vec{V_4} \geq_{\mathbf{w}} \vec{V_3}$ must satisfy $f_4(w) \in_4 f_4(z) \to f_4(w) \in_4 f_4(x)$) must satisfy $f_3(z) \in_3 f_3(y)$.

[13] Specifically, by Simple Comprehension it's possible that the otherwise unused predicate H applies to exactly those a such that $set_1(a) \wedge [(\exists b)(a \in_1 b \wedge b \in_1 f_1(x))$. So by our characterization of V_2 as containing one more layer of sets there is a unique k which contains all and only the a satisfying the condition above.

[14] To show this, consider an arbitrary situation (holding $\vec{V_1}, \vec{V_2}$ fixed) in which $(\forall k)(C(k) \to k \in_2 f_2(x))$. By the fact that $\vec{V_2} \geq_{\mathbf{y}} \vec{V_1}$ we have $(\forall k)(k \in_2 f_2(x) \to k \in_1 f_1(x))$. Then every object satisfying C is available at a level below the level

Informally, this says: it's logically necessary (given the facts about V_2) that if C only applies to objects in $f_2(x)$ then there is some set_2 in $f_2(y)$ which has exactly the objects satisfying C as elements.

Now, it remains to consider an arbitrary situation (holding the facts about our $\vec{V_2}$ fixed) in which $\vec{V_3} \geq_{\mathbf{z}} \vec{V_2} \wedge \Box_{\vec{V_3}}[V_4 \geq_{\mathbf{w}} \vec{V_3} \rightarrow (f_4(w) \in_4 f_4(z) \rightarrow f_4(w) \in_4 f_4(x))]$ (call this hypothesis α) and show that $f_3(z) \in_3 f_2(y)$. From the $\Box_{\vec{V_3}}$ claim in our hypothesis, we can deduce that $f_3(z)$ is a subset of $f_3(x)$ in the sense of V_3 [15]. And because $\vec{V_3} \geq_{\mathbf{z}} \vec{V_2}$, we can further deduce that everything which is $\in_3 f_3(z)$ is also $\in_2 f_2(x)$ [16]. In this way, the elements of $f_3(z)$ correspond to a logically possible subset of $f_2(x)$.

Since any V_3 extending V_2 can't add any elements to $f_2(x)$, it is straightforward to verify that $f_3(z) \in_3 f_3(y)$.[17]

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where $f_2(x) = f_1(x)$ first occurs. Thus there is a set_1 with exactly these elements, call it k. Now by our characterization of $f_2(y)$ as containing exactly those k' such that $set_1(k') \wedge [(\forall k'')(k'' \in_1 k' \rightarrow k'' \in_1 f_1(x))$ we can deduce that $k' \in_2 f_2(y)$. Thus we have $(\exists k')(k' \in_2 f_2(y) \wedge (\forall k)[C(k) \leftrightarrow k \in_2 k'])$ as desired.

[15][this is way too long...you just should say something like..this follows by blah] Suppose for contradiction that $(\exists k'')(k'' \in_3 f_3(z) \wedge \neg k'' \in_3 f_3(x))$. Then (by simplified choice and various applications of simple comprehension combined as per the Multiple Definitions Lemma) it is $\Diamond_{\vec{V_3}}$ that $V_4 \geq_{\mathbf{w}} \vec{V_3}$ with $V_4 =_{set} V_3$ and $f_4(w)$ applies to a unique object k'' such that k" $\in_3 f_3(z) \wedge \neg k'' \in_3 f_3(x)$. In this $\Diamond_{\vec{V_3}}$ context, it must also be true that that $f_4(w) \in_4 f_4(z) \wedge \neg f_4(w) \in_4 f_4(x)$.

But the logical possibility of such a scenario (holding fixed the facts about $\vec{V_3}$) contradicts our prior assumption $\Box_{\vec{V_3}}[V_4 \geq_{\mathbf{w}} \vec{V_3} \rightarrow (f_4(w) \in_4 f_4(z) \rightarrow f_4(w) \in_4 f_4(x))]$.

Thus $f_3(z)$ is a subset of $f_3(x)$ (in the sense of V_3).

[16]Any $k \in_2 f_3(z)$ must also be in $\in_3 f_3(z)$ hence in $\in_3 f_3(x) = f_2(x)$ hence $\in_2 f_2(x)$

[17]We can import the fact that $f_2(y)$ contains all logically possible subsets of $f_2(x)$ in the sense of V_2, since this claim is content restricted to $\vec{V_2}$. By Simple Comprehension, it is possible (holding fixed the facts about $\vec{V_2}, \vec{V_3}$ and hence the facts about $f_2(y)$ and $f_3(z)$ just proved above) for C to apply to exactly the elements of $\in_3 f_3(z)$. Because $(\forall k)[k \in_3 f_3(z) \rightarrow k \in_2 f_2(x)])$ remains true in this context, it follows that that C only applies to objects $\in_2 f_2(x)$. Because our characterization of $f_2(y)$ remains true in this context (note that it is content restricted to $\vec{V_2}$), we know that (necessarily, given V_2 facts) if C only applies to objects $\in_2 f_2(x)$ there is some set_2, k, such that $k \in_2 f_2(y)$ whose elements are exactly the objects satisfying C. Thus there is some set_2 which has exactly the same elements as $f_3(z)$. Now, by the thinness/extensionality requirement built into $\vec{V_3} \geq_z V_2$, we know that $f_3(z)$ = this set_2, so we have $f_3(z) \in_2 f_2(y)$, and hence

From this proof of $\Box_{V_2}(\alpha \to f_3(z) \in_3 f_3(y))$, the desired conclusion follows straightforwardly. $\Box$

Proposition 12.1.7 (Choice)**.** "$\forall x\,[\varnothing \notin x \to \exists f\colon x \to \bigcup x \quad \forall a \in x\,(f(a) \in x)]$"

Writing out almost all the abbreviations, and applying FOL to this yields:
$(\forall x)[(\forall y)(y \in x \to (\exists z)(z \in y)) \to$
$(\exists f)(\forall a)[a \in x \to \exists y(\langle a, y\rangle \in f \wedge (\forall y')[\langle a, y'\rangle \in f \to y = y'])]]$

thus it gets translated as something with the following form [inc: footnote that I'm not expanding out the brackets for pairing?]:

[fix linebreaks]

$\Box[\mathscr{V}(\vec{V_0}) \wedge V_1 \geq_{\boldsymbol{x}} \vec{V_0} \to [$

$\Box_{\vec{V_1}}(V_2 \geq_{\boldsymbol{y}} \vec{V_1} \wedge f_2(y) \in_2 f_2(x) \to \Diamond_{\vec{V_2}}[V_3 \geq_{\boldsymbol{z}} \vec{V_2} \wedge f(z) \in f(y)]) \to$

$\Diamond_{\vec{V_1}}(V_2 \geq_{\boldsymbol{f}} \vec{V_1} \wedge t_2((\forall a)[a \in x \to \exists y(\langle a, y\rangle \in f \wedge (\forall y')[\langle a, y'\rangle \in f \to y = y'])]))$

So it says: if V_1 assigns $f_1(x)$ to something which doesn't contain the empty set[18]*, then one can have an extending V_2, f_2 which assigns $f_2(f)$ to a set which codes up a choice function for $f_1(x)$*[19]*.*

Proof. Unsurprisingly, we will use an instance of the Choice Axiom Schema to prove this claim.

Consider an arbitrary situation in which $\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\mathbf{x}} \vec{V_0}$.

Now suppose that the antecedent of the conditional we need to prove. That is, suppose that $\Box_{\vec{V_1}}(V_2 \geq_{\mathbf{y}} \vec{V_1} \wedge f_2(y) \in_2 f_2(x) \to \Diamond_{\vec{V_2}}[\vec{V_3} \geq_{\mathbf{z}} \vec{V_2} \wedge f(z) \in f_3(z) \in_3 f_3(y)$.

[18] in the sense that for any extending V_2, f_2 assigning y to something in $f_1(x)$ there could be a V_3, f_3 assigning z to something in $f_2(y)$)

[19] in the sense that for any extension V_3, f_3 assigning a to something in $f_1(x)$ makes $t_3(\exists y(\langle a, y\rangle \in f \wedge (\forall y')[\langle a, y'\rangle \in f \to y = y'])])$ come out true

$f(y)]$).

Our first step will be to deduce from the above assumption that the empty set is not an element of $f_1(x)$, (i.e. $(\forall k)k \in_1 f_1(x) \rightarrow \exists k' k' \in_1 k$). We will argue by contradiction.

If an empty set were in $f_1(x)$, then it would be possible (holding fixed the facts about V_1) to have an extending V_2 where $f_2(y)$ is this empty set, hence it is impossible for there to be an extending V_3 where $f_3(z) \in_3 f_3(y)$. But this contradicts the $\Box_{V_1}$ assumption above. [20]

Thus we know that the empty set is not in $f_1(x)$. Now we will (unsurprisingly!) use the Choice Axiom in my formal system to construct a suitable logically possible V_2, f_2, and then show it behaves as desired. By three applications of the One More Layer Lemma, we can have a V_2 which adds three layers of classes to V_1. By Simple Comprehension, it is possible to have an index property I apply to exactly the elements of $f_1(x)$ and a relation R (which we intend to apply Choice to) which applies to exactly pairs a, b consisting of an element $a \in_1 f_1(x)$ and $b \in_1 a$. By an application of Choice to R (importing the fact that $f_1(x)$ does not contain an empty set), we can conclude it is possible that $\hat{R}(a, b)$ associates each a in $f_1(x)$ with a unique b in a. By the Multiple Definitions Lemma we can put all these stipulations together, and then enter a single $\Diamond_{V_1}$ context in which all the characterizations of $V_2, I, R, \hat{R}$ above remain true.

[20]More pediantically, suppose an empty set were in $f_1(x)$. Then it would be $\Diamond_{V_1}$ to have $\vec{V_2} \geq_y \vec{V_1}$, where $V_1 =_{set} V_2$ and $f_2(y)$ is the empty set (in the sense of $\in_1$). Since $f_2(y) \in_1 f_1(x)$, we have $set_1(f_2(y))$ and hence this $f_2(y)$ is an empty set in the sense of $\in_2$ as well, i.e., $\neg(\exists k)(k \in_2 f_2(y))$. So we also have $\neg \Diamond_{\vec{V_2}} [V_3 \geq_{\mathbf{z}} \vec{V_2} \wedge f_3(z) \in_3 f_3(y)]$, since any such $f_3(z) \in_3 f_3(y) = f_2(y)$ would have to be $\in_2 f_2(y)$. Thus we get the possibility of a scenario is ruled out by the $\Box_{V_1}$ assumption above.

Now we can show (laboriously but straightforwardly) that this V_2 contains a set_2 which is the graph of the the choice function $\hat{R}$ specified above. (With a suitable use of Wrapping Trick to mimic $\forall I$ arguments involving modality) we can note that for each $b \in_2 f_1(x)$ there is a c such that $\hat{R}(b,c)$ within V_1. Then we can exploit the fact that V_2 contains three layers of classes over V_1 to show that it contains a pair set $\langle b,c\rangle$, and a set_2 which collects together all such pairs.[21]

Finally, it remains to check that this assignment $f_2(f)$ ensures the truth of $t_2((\forall a)[a \in x \to \exists y(\langle a,y\rangle \in f \wedge (\forall y')[\langle a,y'\rangle \in f \to y = y'])])$. This is somewhat laborious, but we can do it via exactly the same technique demonstrated in the simpler proofs above. Specifically, we argue that all extending V_i, f_i which satisfy relevant antecedents must assign variables to objects at or below $f_2(f)$ and/or $f_2(x)$ (hence to objects in V_2), and then exploit the fact that $f_2(f)$ is the graph of a choice function for $f_2(x)$ [in the sense restricted to V_2]. This completes our Inn $\diamond$ argument that a suitable extending V_2, f_2 is possible. $\square$

[21] More pedantically: by our characterization of V_2, there is (one layer above V_1) a $w = \{c\}, w' = \{b\}\ w'' = \{b,c\}$, and hence (two layers above V_1) a $w'' = \langle b,c\rangle$. Since this is true for each b in the domain of $\hat{R}$, there will be (three layers above V_1) a set_2 which is the graph of $\hat{R}$ i.e., $\{\langle b,c\rangle$ such that $\hat{R}(b,c)\}$.Consider applying Simple Choice to specify the application a property K $(\forall x)[K(x) \leftrightarrow \exists b\exists c(\hat{R}(b,c) \wedge \exists w\exists w'\exists w'' w = \{c\} \wedge w' = \{b\} \wedge w'' = \{b,c\}]$ (with all abbreviations written out in the usual way). By the reasoning above, there will be for each b,c such that $\hat{R}(b,c)$ a corresponding element of K. Also (again, by the reasoning above) all these elements will occur below the last layer of V_2, so by our construction of V_2 there will be a set_2 whose elements are exactly those in the extension of K. By our construction of $\hat{R}$, this set_2 is the graph of a choice function for $f_1(x)$ (i.e., it contains, for each $b \in_2 f_2(x)$, exactly one set of the form $\langle b,c\rangle$, with c such that $b \in_2 c$). So, (by the multiple definitions lemma and ignoring) it is $\diamond_{\vec{V_1}}$ to have $\vec{V_2} \geq_f \vec{V_1}$ with $f_2(f)$ the graph of a choice function for $f_1(x) = f_2(x)$. [in the sense restricted to V_2]

12.2 Comprehension

Proposition 12.2.1. ***Comprehension*** *"Let $\phi(x, w_1, \ldots, w_n)$ be a formula in the language of ZFC with free variables $x, w_1, \ldots, w_n$. Then:*

$\forall z \forall w_1 \forall w_2 \ldots \forall w_n \exists y \forall x[x \in y \Leftrightarrow (x \in z \wedge \phi)]$."

Translating and then applying $\Box$ simplification yields: [fix double subscripts] $\Box[\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\boldsymbol{z}} \vec{V_0} \wedge \vec{V_2} \geq_{w_1} \vec{V_1}\vec{V_1} \wedge \ldots V_{n+1} \geq_{w_n} V_n \rightarrow \Diamond_{V_{n+1}}[V_{n+2} \geq_{\boldsymbol{y}} V_{n+1} \wedge \Box_{V_{n+2}}(V_{n+3} \geq_x V_{n+2} \rightarrow [f_{n+3}(x) \in_{n+3} f_{n+3}(y) \leftrightarrow f_{n+3}(x) \in_{n+3} f_{n+3}(z) \wedge t_{n+3}(\phi)])]$

This says approximately the following. Fix assignments for for $z, w_1, \ldots, w_n$ from set_{n+1} within some V_{n+1}. It's logically possible to have an extending $V_{n+2} \geq_y V_{n+1}$ which assigns y to a set which collects together exactly those x in $f_{n+2}(z)$ such that that any extending V_{n+3}, f_{n+3} which assigns $f_{n+3}(x)$ to one of these x must make $t_{n+3}(\phi(z, w_1, \ldots, w_n, x))$ true.

Proof. Suppose that $\vec{V_0}...\vec{V_{n+1}}$ are as above.

Our first task will be to establish the logical possibility of a suitable $f_{n+2}(y)$ and V_{n+2}. Let t_{n+3**} represent the result of replacing all occurrences of relations in V_{n+3}, f_{n+3} in t_{n+3}, with occurrences of relations in V_*, f_*.[22] By using the Modal Comprehension Schema, we can show that a predicate P could apply to exactly $x \in_{n+1} f_{n+1}(z)$ with the following modal property: there could be an extension $\vec{V_{n+3}}* \geq_x \vec{V_{n+1}}$ such that $f_{n+3} * (x) = x$ and $t_{n+3}{*}{*}(\phi)$ comes out true (note that $t_{n+3}{*}{*}(\phi)$ makes mention of $f_{n+3}{*}(x)$).[23]

[22] So, as in the proof of lemma **??** occurrences of set_{n+3} are replaced with occurrences of set_*, but occurrences of set_{n+4} inside $\Box$es and $\Diamond$s are unchanged.

[23] To see why this is true more formally, consider the formula asserting that it is logically possible that f_{n+3} matches f_{n+1} everywhere but on the variable x which it takes the unique value satisfying Q (where Q is the predicate from the Modal Comprehension axiom) and

These will turn out to be exactly the objects we want our set $f_{n+2}(y)$ to collect. By the fact that $\mathscr{V}(V_{n+1})$, there's a set_{n+1} y, whose elements are exactly those those satisfying P. This will be our choice for $f_{n+2}(y)$ and we will let V_{n+2} be equal to V_{n+1}.

Now it remains to check that V_{n+2}, f_{n+2} behaves as desired. We need to show that $\Box_{V_{n+2}}$ if $\vec{V_3} \geq_x \vec{V_2}$, assigns $f_{n+3}(x)$ to something in $f_{n+3}(y)$ iff it satisfies $t_{n+3}(\phi(z, w_1, \ldots, w_n, x))$. By Ign we know that if there could be a counterexample to the claim above, then there could be a counterexample which holds fixed V_{n+1}, P as well as V_{n+2}. So consider an arbitrary scenario (holding fixed V_{n+1}, P, V_{n+2}) in which $\vec{V_{n+3}} \geq_{\mathbf{x}} \vec{V_{n+2}}$. It suffices to show that $f_{n+3}(x) \in_{n+3} f_{n+3}(y) \leftrightarrow f_{n+3}(x) \in_{n+3} f_{n+3}(z) \wedge t_{n+3}(\phi)$ in this scenario. There are two directions to check.

$\rightarrow$ Suppose f_{n+3} assigns x to something in $f_{n+3}(y)$ (our supposed comprehension set). Then our characterization of $f_{n+2}(y)$[24] implies that this object is in $f_{n+1}(z)$ (the set we are comprehending over). So we have $f_{n+3}(x) \in_{n+3} f_{n+3}(z)$ immediately. Now we need $t_{n+3}(\phi)$. Our characterization of $f_{n+2}(y)$ also says [via the wrapping trick for mimicing quantifying in] that because $f_{n+3}(x) \in_{n+2} f_{n+2}(y)$, it is possible (holding fixed V_1) for an extension $\vec{V_{n+3}}* \geq_x \vec{V_{n+1}}$ which assigns $f_{n+3}*(x) =$ to (an object in structurally the same position w.r.t. V_{n+1}, f_{n+1} as our) $f_{n+3}(x)$ to make $t_{n+3}**(\phi)$ true.

$t_{n+3}(\phi)$ comes out true.

$\Diamond_{\vec{V_{n+1}}, Q}[(\forall r \neq \ulcorner x \urcorner) f_{n+3}(r) = f_{n+1}(r) \wedge$
$(\forall q)(f_{n+3}(\ulcorner x \urcorner) = q \leftrightarrow Q(q)) \wedge$
$t_{n+3}(\phi)]$

This formula can be plugged directly into the Modal Comprehension axiom, and we can derive that the resulting property P applies to all and only those $x \in_{n+1} f_{n+1}(z)$ with the property informally described above.

[24] This must remain true in our current context because it is content-restricted to V_{n+1}, V_{n+2}.

We can infer that the same scenario is possible while holding fixed the $\vec{V_{n+1}}, \vec{V_{n+2}}$ facts as well, by Ignoring.[25]. So we can enter this $\Diamond_{\vec{V_{n+1}},\vec{V_{n+2}},\vec{V_{n+3}}}$ context, and import all previously established facts about V_{n+2} and V_{n+3}. Now it remains to use the Translation Lemma to go from $t_{n+3} * *(\phi)$ to $t_{n+3}(\phi)$.

The trick will be to cook up a $V_{n+1}, f_{n+1}*$ which agrees with t_{n+3} and $t_{n+3} * *$ on the assignment of x and all other variables free in ψ, and then use a version of the Translation Lemma to go from $t_{n+3} * *(\phi)$ to $t_{n+1@}(\phi)$ to $t_{n+3}(\phi)$. For, note that we have $V_{n+3} \geq V_{n+1}$ and $V_{n+3}* \geq V_{n+1}$ and that f_{n+3} agrees with $f_{n+3}*$ in assigning all variables free in ϕ to objects in V_1: y is not free in ϕ, f_{n+3} agrees with $f*_{n+3}$ on the assignment of x to something $\in f_{n+1}(x)$ hence in V_1 by construction, and on all other free variables $w_1 ... w_n$ in ϕ both f_{n+3} and $f_{n+3}*$ agree with f_{n+1}). Thus if we use modal comprehension to let $f_{n+1}@ = f_{n+1}$ everywhere except in assigning x to $f_{n+3}(x) = f_{n+3*}(x)$, inside this $\Diamond_{\vec{V_{n+1}},f*,\vec{V_{n+2}},\vec{V_{n+2}}}$ scenario we will have $\mathscr{V}(V_{n+1}, f_{@})$. Thus we will try to use the Translation lemma to get $t_{n+3} * *(\phi) \leftrightarrow t_{n+1@}(\phi) \leftrightarrow t_{n+3}(\phi)$, as desired.

Once we have done this, we are finished. For, from the fact that $t_{n+3}(\phi)$ in the above $\Diamond_{\vec{V_{n+1}},f*,\vec{V_{n+2}},\vec{V_{n+3}}}$ scenario, we can infer that it holds in our original scenario as well.

[Now it just remains to deal with the wrinkle that (as before) the Translation Lemma doesn't directly say anything about $t_{n+3} * *(\phi)$ or $t_{n+1@}(\phi)$. However, we can use the $\Box$ relabling to get what we need as before. First re-

[25]This inference is permitted because the inside of the $\Diamond_{\vec{V_{n+1}}}$ claim is content restricted to $\vec{V_{n+1}}, \vec{V_{n+3}}*$ and there is no overlap between $V_{n+3}*$ and V_{n+2}, V_{n+3}

place all instances of f_{n+1} with $f_{n+1@}$. Then replace all instances of V_{n+3}, f_{n+3} with corresponding V*,f*, but notice that there may be some collateral damage. Any mentions of V_{n+3} within $t_{n+1}(\phi)$ will be replaced, so we have more work to do if ϕ it contains any quantifiers nested 2 deep. Fortunately, however, we can undo this damage, by entering into the t_{n+2} contexts housing each instance of such nested quantification which got changed. The $\Box$ and $\Diamond$ relabeling let us derive that $\Box t_{n+2}(\rho) \leftrightarrow t_{n+2}(\rho)[t_{n+3}/t_{n+3} * *]$ (or the corresponding $\Diamond$ claim in each of these contexts, and hence to fix all such collateral damage.] [FIX wording]

$\leftarrow$ Conversely, suppose f_{n+3} assigns x to something in $f_{n+3}(z)$ (the set being comprehended over) and that $t_{n+3}(\phi)$. By our characterization of $f_{n+2}(y)$, we can show that the relevant object is also in $f_{n+3}(y)$ if we establish two things. First, we need the object is ϵ_1 $f_{n+1}(z)$. This follows immediately, because $\vec{V_{n+3}} \geq_x \vec{V_{n+2}} \geq_y \vec{V_{n+1}}$.

Second, we need to show that it is $\Diamond_{V_{n+1}}$ to have $\vec{V_{n+3}}* \geq_x \vec{V_{n+1}}$ such that [again, speaking loosely and using the Wrapping Trick to mimic quantifying in] $f_{n+3} * (x) =$ this $f_{n+3}(x)$ and $t_{n+3} * *(\phi)$.

I will prove this by proving the stronger corresponding $\Diamond_{\vec{V_{n+1}},\vec{V_{n+2}},\vec{V_{n+3}}}$, claim. By assumption, we have $t_{n+3}(\phi)$. By simple comprehension, it is $\Diamond_{\vec{V_{n+1}},\vec{V_{n+2}},\vec{V_{n+3}}}$ to have $t_{n+3}(\phi)$ remain true while $V_{n+3}* =_{set} V_{n+3}$ and $f_{n+3}*$ agrees with f_{n+3} everywhere, except that $f_{n+3}(y) = f_{n+1}(y)$ it agrees with f_{n+1} on the assignment of y. Now (just as above) we can use the generalized Translation Lemma to go from the fact that V_{n+3} and this V_{n+3*} both extend V_{n+1} and agree in assigning all variables free in ϕ (because y is not free in ϕ) to objects in V_{n+1} to the conclusion that $t_{n+3}(\phi) \leftrightarrow t_{n+3} * *(\phi)$. This gives us $t_{n+3}(\phi)$, as desired.

Combining the $\rightarrow$ and $\leftarrow$ arguments above complete the desired proof that $f_{n+3}(x) \in f_{n+3}(y) \leftrightarrow f_{n+3}(x) \in_{n+3} f_{n+3}(z) \wedge t_{n+3}(\phi)$.

$\square$

12.3 Infinity

Proposition 12.3.1. ***Infinity*** *"$\exists x\, [\varnothing \in x \wedge \forall y(y \in x \rightarrow S(y) \in x)]\,.$"*

where $S(x)$ is $x \cup \{x\}$.

Let

$$\ulcorner \varnothing \in f_1(x) \urcorner = \Diamond_{\vec{V}_1} (\vec{V}_2 \geq_{\boldsymbol{e}} \vec{V}_1 \wedge \Box_{\vec{V}_2}[\vec{V}_3 \geq_{\boldsymbol{z}} \vec{V}_2 \rightarrow \neg f_3(z) \in_3 f_3(e)] \wedge f_2(e) \in_2 f_2(x))$$

$$\ulcorner S(f_2(y)) \in f_2(x) \urcorner = \Diamond_{\vec{V}_2} (\vec{V}_3 \geq_{\boldsymbol{s}} \vec{V}_2 \wedge \Box_{\vec{V}_3}[V_4 \geq_{\boldsymbol{z}} \vec{V}_3 \rightarrow f_4(z) \in_4 f_4(s) \leftrightarrow$$
$$f_4(z) \in f_4(y) \vee f_4(z) = f_4(y)] \wedge f_3(s) \in_2 f_3(x)])$$

Using these suggestively named components, the translation of infinity can be written as:

$$\Box(\mathscr{V}(V_0) \rightarrow \Diamond_{\vec{V}_0}[\vec{V}_1 \geq_{\boldsymbol{x}} \vec{V}_0 \wedge \ulcorner \varnothing \in f_1(x) \urcorner$$

$$\wedge \Box_{\vec{V}_1} (\vec{V}_2 \geq_{\boldsymbol{y}} \vec{V}_1 \wedge f_2(y) \in_2 f_2(x) \rightarrow \ulcorner S(f_2(y)) \in f_2(x) \urcorner)]$$

Proof. Consider an arbitrary scenario in which $\mathscr{V}(V_0)$ holds. On this assumption, we can show that the suggestive names we used for parts of the translation above are accurate: if $\vec{V}_1 \geq_{\mathbf{x}} \vec{V}_0$ then $\ulcorner \varnothing \in f_1(x) \urcorner$ holds if $\varnothing \in_1 f_1(x)$ and if, furthermore, $\vec{V}_2 \geq_{\mathbf{y}} \vec{V}_1$, then $\ulcorner S(f_2(y)) \in f_2(x) \urcorner$ holds if $S(f_2(y)) \in_2 f_2(x)$.

For example, note that if $\varnothing \in_1 f_1(x)$ then it's possible to have $V_2 =_{set} V_1$ and f_2 equal to f_1 everywhere except at e and $f_2(e) = \varnothing$. It's thus necessary, holding V_2, f_2 fixed, that if $\vec{V_3} \geq_{\mathbf{z}} \vec{V_2}$ then $\neg f_3(z) \in_3 f_3(e)$ as $f_3(e) = f_2(e) = \varnothing$. This establishes the claim about $\ulcorner \varnothing \in f_1(x) \urcorner$. Similar elementary reasoning establishes the above claim about $\ulcorner S(f_2(y)) \in f_2(x) \urcorner$.

I will establish the logical possibility claim that we need, by arguing as follows. By the Infinite Well-Ordering Lemma (proved in section 9.0.1) there can be a an infinite well ordering $\omega, \leq$ which contains only successor stages. By the Fleshing Out Lemma (C.7), it is logically possible to have an initial segment V_ω, whose ordinals $ord_\omega, \leq_\omega$ are isomorphic to $\omega, \leq$. Using the Recursive Definition Lemma (proved in section B.1.1), we define a function F from ω to V_ω with $F(0) = \varnothing$ and $F(n+1) = S(F(n))$ and then use induction establish the domain of F is ω. By the definition of ω, for each $n \in \omega$ there is an $n+1 \in \omega$ such that $F(n+1) = S(F(n))$. We then establish the possibility of an initial segment $V_{\omega+1}$ containing an extra layer of sets over those in V_ω and thus containing a set x whose members are exactly the elements in the range of F. The theorem follows by observing that letting V_1 be $V_{\omega+1}$ makes the sentence true.

Now let us go into details. By the Infinite Well-Ordering Lemma, we can have a well-ordering $\omega, <$ without a maximal element where every element satisfying ω is either 0 or a successor.

By the Fleshing Out Lemma we can infer $\Diamond_{\omega,<} \mathscr{V}(\text{set}_\omega, \in_\omega, @_\omega, \omega, <)$. Assume V_ω is the tuple of relations having these properties. Next we can use the Recursive Definition Lemma to establish the logical possibility of a two place relation $F(o, z)$ between objects satisfying ω and set_ω adopting the functional abbreviation $F(o) = z$ for clarity

$$F(o) = z \iff \begin{cases} o = 0 \land z = \varnothing \\ \qquad \lor \\ o = n + 1 \land z = F(n) \cup \{F(n)\} \end{cases}$$

Where $\varnothing$ is the element in set$_\omega$ containing no other elements under $\in_\omega$ and $F(n) \cup \{F(n)\}$ is the element in set$_\omega$ whose elements are exactly the members of $F(n)$ and $F(n)$.

[We can check that the premises needed for the Recursive Definition Lemma are satisfied, as follows].Clearly, it is logically necessary (given the facts about $\omega, \leq$ and V_ω) that a unique object satisfies $x = \varnothing$ in V_ω. And for n s.t. $\neg n = 0$ and $\omega(n)$, we know that n is a successor ordinal (so there is an m such t $n = m + 1$) by our characterization of $\omega, \leq$. Thus [it is logically necessary (given the facts about $\omega, \leq$ and V_ω) that] if F is defined and functional below n, we have the existence of an x such that $(\exists m)[n = m + 1 \land x = S(F(m))]$ because ord_ω include a successor ordinal for every ordinal which it contains (and hence a stage above every stage it contains) and $S(F(m))$ must occur a stage above wherever $F(m)$ occurs (by the fact that $\mathscr{V}(V_\omega)$ and our definition S). We the have uniqueness of this x by the extensionality of the set_ω and the definition of S.

Now by the One More Layer Lemma (proved in section C.4) we can infer the possibility of $V_{\omega+1}$ extending V_ω and adding a single layer of classes. Now all the objects in the image of F are sets in V_ω. Thus $V_{\omega+1}$ contains a set I whose members are exactly those elements of V_ω such that $(\exists o)(\omega(o) \land F(o) =$

x). This set contains $\varnothing$ (a set which has no elements in the sense of $V_{\omega+1}$ and hence also none in the sense relevant to $V_{\omega+2}$) and is closed under application of S.

Lastly, it remains to show that we can find a set like I in an initial segment extending $\vec{V_0}$. By the Hierarchy Extending Lemma (proved in C.5) if is logically possible to have an extension V_1 of $\vec{V_0}$, such that Z isomorphically maps from $V_{\omega+1}$ to an initial segment of V_1. It is a straightforward, if somewhat tedious, process to verify that the image of our I under Z also behaves like a suitable infinite set: it contains an object $\varnothing$ which has no elements in the sense of V_1, and contains the the successor of every set_1 it contains.

To complete the proof, note that we can let $f_1(x)$ be $Z(I)$. Clearly $\ulcorner \varnothing \in f_1(x) \urcorner$ holds in this case and if $\vec{V_2} \geq_{\mathbf{y}} \vec{V_1} \wedge f_2(y) \in_2 f_2(x)$ then $S(f_2(y)) \in f_2(x)$ so $\ulcorner\ S(f_2(y)) \in f_2(x) \urcorner$ holds.

□

12.4 Replacement

Proposition 12.4.1. ***Replacement***

"The axiom schema of replacement asserts that the image of a set under any definable function will also fall inside a set.

Formally, let ϕ be any formula in the language of ZFC whose free variables are among $x, y, A, w_1, \ldots, w_n$, so that in particular B is not free in ϕ. Then:

$$\forall A \forall w_1 \forall w_2 \ldots \forall w_n \big[\forall x (x \in A \to \exists! y\, \phi) \to \exists B \forall x \big(x \in A \to \exists y (y \in B \wedge \phi) \big) \big].$$

In other words, if the relation ϕ represents a definable function f, A represents its domain, and f(x) is a set for every x in that domain, then the range of f is a subset of some set B."

Instances of this schema have a translation with the form

$\Box[\mathscr{V}(\vec{V}_0) \Box (\vec{V}_1 \geq_{\boldsymbol{a}} \vec{V}_0 \rightarrow \Box[\vec{V}_2 \geq_{\boldsymbol{w1}} \vec{V}_1 \ldots \Box V_{n+1} \geq_{\boldsymbol{wn}} V_n \rightarrow (\alpha \rightarrow \beta)]...])]$

which, by $\Box$ simplification, becomes:

$\Box[\mathscr{V}(\vec{V}_0) \wedge \vec{V}_1 \geq_{\boldsymbol{a}} \vec{V}_0 \wedge \vec{V}_2 \geq_{\boldsymbol{w1}} \vec{V}_1 \ldots \Box V_{n+1} \geq_{\boldsymbol{wn}} V_n \rightarrow (\alpha \rightarrow \beta)]]$

where:

- $\alpha = \Box_{V_{n+1}}(V_{n+2} \geq_{\boldsymbol{x}} V_{n+1} \wedge f_{n+2}(x) \in_{n+2} f_{n+2}(a) \rightarrow \Diamond_{V_{n+2}}[V_{n+3} \geq_{\boldsymbol{y}} V_{n+2} \wedge \boldsymbol{t}_{n+3}(\phi(w_1, \ldots w_n, x, y)) \wedge \Box_{V_{n+3}} V_{n+4} \geq_{\boldsymbol{z}} V_{n+3} \wedge \boldsymbol{t}_{n+4}(\phi(w_1, \ldots w_n, x, z)) \rightarrow f_{n+4}(y) = f_{n+4}(z))])$
- $\beta = \Diamond_{V_{n+1}}(V_{n+2} \geq_{\boldsymbol{b}} V_{n+1} \wedge \Box_{V_{n+2}}[V_{n+3} \geq_{\boldsymbol{x}} V_{n+2} \wedge f_{n+3}(x) \in_{n+3} f_{n+3}(a) \rightarrow \Diamond V_{n+4} \geq_{\boldsymbol{y}} V_{n+3} \wedge f(y) \in f(b) \wedge \boldsymbol{t}_4(\phi(w_1, \ldots w_n, x, y))])$

Proof Sketch:

In essence, the translation of the Replacement Schema's antecedent [α] asserts that for every x in a there is a logically possible [it would be possible to have an] initial segment V_x and an element y of that segment such that y is the unique solution to $t(\phi(x, y))$.

And the translation of Replacement's consequent [β] demands that we produce a *single* logically possible initial segment [(call it V_Σ)] containing a y for every x in a (technically containing a set b containing all such y's but that is fixed by one more layer)[satisfying $t_\Sigma(\phi(x, y))$.

Now, the Translation Lemma tells us that if $t_x(\phi(x, y))$ holds in some V_x, then it holds in any extension of V_x which preserves the assignment of x

and y and all the other free variables in ϕ. Thus, it is enough to demonstrate the possibility of some V_Σ extending each V_x.

To achieve this end, we first invoke Combinatorial Replacement to [get (the logical possibility of) simultanioulsy having a collection of hierarchies V_x parametrized to each $x \in_n f_n(a)$] parameterized the V_x by x and then invoke the Mass Hierarchy Combining Lemma (proved in C.6) to (essentially) get a single initial segment extending them all. Adding one extra layer of sets on top of that is enough to produce the desired set B.

Proof. Consider an arbitrary situation with $\vec{V_0}...\vec{V_{n+1}}$ as above. Assume that our translation of the antecedent to replacement, α, is true.

Constructing the $V_x s$ with Combinatorial Replacement

[fill in missing "vec"s as per new notion]

Our first step will be to use the Combinatorial Replacement Schema to establish that a single scenario could associate each $x \in_{n+1} f_{n+1}(a)$ with a corresponding initial segment V_x extending V_{n+1} and containing a witness y satisfying $t(\phi(x,y))$.

Our assumption α guarantees that for any $\vec{V_{n+2}}$ extending V_{n+1} which assigns x so as to satisfy $t_{n+2}(f(x) \in f(a))$, there can be a $\vec{V_{n+3}}$ extending $\vec{V_{n+2}}$ and which assigns y so that $t_{n+3}(\phi)$ comes out true (where $t_{n+3}(\phi)$ implicitly refers to x and y via f_{n+3}).

It is logically possible that I applies to exactly those objects which are $\in_{n+1} f_{n+1}(a)$. Entering this $\Diamond_{\vec{V_{n+1}}}$ scenario, α will remain true. And it is easy to see that α implies the following modal claim. For any way P could 'select' a single object satisfying I (and hence for every possible choice of $f_{n+2}(x)$ on which $t_{n+2}((f(x) \in f(a))$ comes out true), there could be an extension V_x which agrees with V_{n+1} on everything but x and y, assigns x to the object selected by P and makes $t_x(\phi)$ come out true.

$$\Box_{V_{n+1}}(\exists! x P(x)) \wedge I(x) \to \Diamond_{V_{n+1},P}[V_x \geq_{x,y} V_{n+1} \wedge (\forall k)(f_x(x) = k \to P(k)) \wedge t(\phi)]$$

This statement is in the form needed to apply the Combinatorial Replacement Axiom Schema. Thus, by instantiating this schema we can derive the corresponding consequent that it is logically possible (holding fixed I, V_{n+1}) for there to simultaniously be a bunch of different $V_{\hat{x}}$ indexed to each of the different objects $\hat{x}$ satisfying I, i.e., to the $\hat{x} \in_{n+1} f_{n+1}(a)$. [More strictly we get that it is possible for there to be a relation [fill in good notation for it here] that codes up the behavior of each $V_{\hat{x}}$]

Constructing V_{n+2}, f_{n+2}

Next we want to argue that one can have an extending V_{n+2} which assigns b to an object that 'gathers up', for each possible assignment of x to something $\hat{x} \in_{n+1} f_{n+1}(a)$, (the images under isomorphism of) the choice for y made by the corresponding $V_{\hat{x}}$ in which $t(\phi(x,y))$ come out true.

First we build a suitable hierarchy of sets. We use the V-Combining Lemma to get a hierarchy of sets V_{Σ}, which has initial segments isomorphic to each of the scattered $V_{\hat{x}}$ described above (under a certain relation Z [check that def of iso only requires that Z behave like an iso when restricted to the relevant pair of objects]). Then we use One More Layer to argue for the logical possibility of extending this hierarchy of sets by one more layer. Finally we use the Hierarchy Extending Lemma to get that this structure is isomorphic to one that extends V_{n+1}.

This structure will be the V_{n+2} in our desired V_{n+2}, f_{n+2}.[26] It contains a set_{n+2} which collects together the set_{n+2} which are in the images of each $f(y)$ chosen by the V_x for $x \in_{n+1} f_{n+1}(a)$ (under the relevant combination of isomorphisms).[27] Thus we can have $\vec{V_{n+2}} \geq_b \vec{V_{n+1}}$ with $f_{n+2}(b)$ as above.

Checking that V_{n+2}, f_{n+2} behaves as intended

Finally, we must show that the V_{n+2}, f_{n+2} we have constructed makes β, the translation of the consequent of the replacement axiom schema true. Consider an arbitrary extension V_{n+3} which assigns x to something $\in_{n+2}$ $f_{n+2}(a)$. We need to show that there can be an extending V_{n+4} which assigns y to something in $f_{n+2}(b)$ and satisfies $t_{n+4}(\phi)$.

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[26] By the V-Combining Lemma, it is logically possible to have a V_Σ, such that each of the hierarchies of objects satisfying $set *_{n+3} (\cdot, k), \in *_{n+3}(\cdot, \cdot, k)$ for some $k \in_{n+1} f_{n+1}(a)$ is isomorphic to an initial segment of this V_Σ via the relation Z. By the Hierarchy Extending Lemma, we could have a $V_{\Sigma *} \geq_{set} V_{n+1}$, such that V_Σ is isomorphic to an initial segment of $V_\Sigma *$ via the relation Z'. Finally by the One More Layer lemma it is possible to have $V_{n+2} \geq_{set} V_{\Sigma *}$ which adds one more layer of sets to $V_{\Sigma *}$.

[27] Specifically we define $f_{n+2}(b)$ as follows:

For each $k \in_{n+1} f_{n+1}(a)$ there is a $k' = f_{3,k} * (y)$ the choice of $f_{n+3}(y)$ within the initial segment associated with k. We want $f_{n+2}(b)$ to be a set which gathers up (the isomorphic images of) all such sets. Specifically, note that each k' above gets taken to something in V_Σ by Z and then to something in $V_\Sigma *$ by Z'. By simple comprehension a property P could apply to exactly those $k*$ in $V_{\Sigma *}$ such that $\exists k \exists k' Z'(Z(f_{3,k} * (y))) = k*$. So by the fact that the sets for our V_{n+2} are generated by adding one more layer of classes to $V_{\Sigma *}$, we know that there is a set_2 with the above property, i.e., a set_{n+2} whose elements are exactly those $k*$ such that $\exists k \exists k' Z'(Z(f_{3,k} * (y))) = k*$. Let $f_{n+2}(b)$ be this set, and otherwise let $f_{n+2} = f_{n+1}$, so that we have $V_{n+2} \geq_b V_n + 1$.

Finally by the Multiple Stipulations Lemma, it is $\Diamond_{\vec{V_{n+1}}}$ to simultaniously have $\vec{V_{n+2}}, \vec{V_\Sigma}, \vec{V_\Sigma} *, Z, Z'$ satisfying all of the successive definitions above.

To do this, we note that we must also have $x \in_{n+1} f_{n+1}(a)$,[28] hence there is some $V_{\hat{x}}$ indexed by x. This $V_{\hat{x}}$ assigns x to $f_3(x)$ and assigns y in such a way as to make $t_{\hat{x}}(\phi(x,y))$ [i.e.$t_{n+3@@}(\phi(x,y))$ in the logically possible scenario where $V_@, f_@$ behaves like $V_{\hat{x}}, f_{\hat{x}}$] true. And this $V_{\hat{x}}$ can be isomorphically mapped to an initial segment of V_{n+2} (by composing the sequence of isomorphisms mentioned above). Thus we can have $\vec{V_{n+4}} \geq_{\mathbf{y}} \vec{V_{n+3}}$ where $V_{n+4} =_{set} V_{n+3}$ and $f_{n+4}(y)$ is the image of $f_{\hat{x}}(y)[f_@]$ under this isomorphism. This choice of $f_{n+4}(y)$ immediately ensures that $t_{n+4}(y \in b)$ is true, by our characterization of $f_{n+3}(b)$.

Furthermore, there is an obvious extended isomorphism between some V_*, f_* (where V_* is an initial segment of V_{n+4}) and $V_{\hat{x}}, f_{\hat{x}}$ [i.e. $V_@, f_@$][29]. Thus by the Isomorphism Lemma we can infer from the fact that $t(\phi(x,y))$ is true in $V_{\hat{x}}, f_{\hat{x}}$ [i.e. the fact that $t_{n+3@@}(\phi(x,y))$] to the claim that it is true in V_*, f_*.[i.e., $t_{n+3**}(\phi(x,y))$]

Finally, we can use (a version of) the Translation Lemma to infer from the truth of $t(\phi(x,y))$ in V_*, f_* [i.e., $t_{n+3**}(\phi(x,y))$] to its truth in V_{n+4}, f_{n+4}. For we have $V_{n+4} \geq V_*$, and we know that $f_{n+4} = f_*$ on all variables free in ϕ as follows. On $w_1 ... w_n$, f_{n+4} agrees with f_{n+1} and so does f_*, by the fact that all the $V_{\hat{x}}, f_{\hat{x}}$ agree with V_{n+1} on these values, and some reasoning involving the Isomorphism Agreement Lemma[30]. On x, we have

[28]since $V_{n+2} \geq V_{n+1}$

[29]The only issue is to blend the isomorphism between hierarchies with the possible isomorphism between different copies of structures satisfying $PA_\diamond$. (Note that the categoricity of $PA_\diamond$ is an immediate correlary of the well ordering comprability lemma)

[30]Consider the isomorphism between initial segments of V_{n+4} induced by restricting the map from $V_{\hat{x}}$ to V_* to the portion of $V_{\hat{x}}$ which is V_{n+1}. The domain of this map contains $f_{\hat{x}}(w_i)$ for each w_i, since $\vec{Vx} \geq_{x,y} \vec{V_{n+1}}$. Since this map must behave the same as the identity automorphism from V_{n+1} to V_{n+1}, it must map each $f_*(w_i) = f_1(w_i)$ to $f_1(w_i)$.

$f_*(x) = f_{\hat{x}}(x)[= f_{@}(x)] = f_{n+4}(x)$, by our choice of which $V_{\hat{x}}$ to consider. And on y [the giant image set we have so arduously constructed] we have $f_*(y) = f_{n+4}(y)$= the isomorpic image of $f_{\hat{x}}(y)$, by our characterizations of $f_{n+4}(y)$ and f_*. Thus applying a version of the translation lemma will let us infer from truth of $t(\phi(x,y))$ in V_*, f_* [i.e., $t_{n+3**}(\phi(x,y))$] to the conclusion that $t_{n+4}(\phi(x,y))$ in the scenario above, as desired.

[The only wrinkle is that, as in previous cases, the Translation Lemma only directly tells us that $\vdash V_{n+4} \geq V_{n+3} \wedge f_{n+1}(v) = f_{n+3}(v) \wedge \ldots \rightarrow (t_{n+4}(\psi) \leftrightarrow t_{n+3}(\psi))$ claim. But, because of the box introduction rule, we also have $\vdash \Box()$ of the claim above. So by applying $\Box$ relabling, we can make the needed substitutions to get $\vdash \Box(V_{n+4} \geq V_{n+3**} \wedge f_{n+1}(v) = f_{n+3*}(v) \wedge \ldots \rightarrow (t_{n+4}(\psi) \leftrightarrow t_{n+3**}(\psi)))$. Finally, inferring from necessity to truth gives us the desired claim.]

Leaving $\Diamond$ contexts and dropping subscripts as needed gives us $\Diamond_{\vec{V_{n+3}}} \vec{V_{n+4}} \geq_{\mathbf{y}} \vec{V_{n+3}} \wedge f(y) \in f(b) \wedge t_{n+4}(\phi(x,y))$ and then β itself.

This gives us the conditional $\alpha \rightarrow \beta$, as desired. Now successively completing $\Box$I arguments and concluding conditional proofs (just as in all the previous cases) gives us the full modal translation of the relevant instance of the ZFC Replacement Schema.

$\Box$