Chapter 11

Vindication of FOL Inference

We are now in a position to begin the main task of this monograph, by justifying and explaining mathematicians' ordinary use of first order logic in set theory, from a potentialist point of view. Note that my potentialist translations of a set theoretic sentence have a very different logical form from the original. Thus, it is not immediately obvious that whenever a set theoretic sentence β is a first order logical consequence of α, then $t(\beta)$ is a genuine logical consequence of $t(\alpha)$. And it is not obvious that set theorists' willingness to move from a sentence that looks like it asserts α to one that looks like it asserts β is justified.

In this section I will show that every first-order logical argument in the language of set theory can be transformed into an argument in the deduction system described above, which takes us from the translation of the premises for this argument to the translation of its conclusion.

11.1 Useful Lemmas

Lemma 11.1.1 (Pointwise Interpratation Tweaking). *If* $\Phi = (\exists x \mid \mathrm{set}(x))\,(\phi(x))$ *is a sentence content restricted to* $V, \rho, \mathcal{L}$ *and* u *is a formal variable in our language of set theory, i.e.,* $\mathbb{N}(\ulcorner u \urcorner)$*, and neither* ρ' *nor* V' *are in* $\mathcal{L}$ *then*

$$(\mathscr{V}(V,\rho) \wedge \Phi) \leftrightarrow \Diamond_{(V,\rho),\mathcal{L}} \left[(V,\rho') \geq_u (V,\rho) \wedge \phi(\rho'(\ulcorner u \urcorner)) \wedge \mathrm{set}\left(\rho'(\ulcorner u \urcorner)\right)\right]$$

Moreover,

$$(\mathscr{V}(V,\rho) \wedge \Phi) \leftrightarrow \Diamond_{(V,\rho),\mathcal{L}} \left[(V',\rho') \geq_u (V,\rho) \wedge \phi(\rho'(\ulcorner u \urcorner)) \wedge \mathrm{set}\left(\rho'(\ulcorner u \urcorner)\right) \wedge V' = V\right]$$

Where $V' = V$ *is understood to abbreviate the claim that the relations* $\in, <$ @ *apply to exactly the same objects as* $\in', <'$ @$'$

Proof. We first prove the primary claim.

($\leftarrow$) Assume

$$\Diamond_{(V,\rho),\mathcal{L}} \left[(V,\rho') \geq_u (V,\rho) \wedge \phi(\rho'(\ulcorner u \urcorner)) \wedge \mathrm{set}\left(\rho'(\ulcorner u \urcorner)\right)\right]$$

. Using proposition 6.2.10 we infer $\Diamond_{(V,\rho),\mathcal{L}} \mathscr{V}(V,\rho) \wedge \Phi$. As this is content restricted to V, ρ the left hand side follows by 6.2.6.

($\rightarrow$) Assume $\mathscr{V}(V,\rho) \wedge \Phi$. By Simplified Choice we can infer

$$\Diamond_{(V,\rho),\mathcal{L}} (\exists w)\,[\phi(w) \wedge \mathrm{set}(w) \wedge P(w) \wedge (\forall z)\,(P(z) \rightarrow w = z)]$$

Now we enter the $\Diamond_{(V,\rho),\mathcal{L}}$ context. Now using proposition Inner Diamond

With Simplification we can use comprehension in this context to infer

$$\rho'(x) = y \overset{\text{def}}{\leftrightarrow} (x = \ulcorner u \urcorner \wedge P(y)) \vee (x \neq \ulcorner u \urcorner \wedge \rho(x) = y)$$

As P is satisfied by a unique element it is immediate that ρ' is a function. It is also immediate that $(V, \rho') \geq_u (V, \rho)$. Finally, as $P(w) \to \phi(w) \wedge \text{set}(w)$ it follows that $\phi(\rho'(\ulcorner u \urcorner)) \wedge \text{set}(\rho'(\ulcorner u \urcorner))$.

For the moreover claim by the primary claim it suffices to show, on the assumptions in the lemma, that

$$\Diamond_{(V,\rho),\mathcal{L}} \left[(V, \rho') \geq_u (V, \rho) \wedge \phi(\rho'(\ulcorner u \urcorner)) \wedge \text{set}\left(\rho'(\ulcorner u \urcorner)\right)\right]$$

$$\leftrightarrow$$

$$\Diamond_{(V,\rho),\mathcal{L}} \left[(V', \rho') \geq_u (V, \rho) \wedge \phi(\rho'(\ulcorner u \urcorner)) \wedge \text{set}\left(\rho'(\ulcorner u \urcorner)\right) \wedge V' = V\right]$$

the $\leftarrow$ direction is trivial as if $V' = V$ and $(V', \rho') \geq_u (V, \rho)$ then $(V, \rho') \geq_u (V, \rho)$.

For the $\to$ direction enter the $\Diamond_{(V,\rho),\mathcal{L}}$ and by Inner Diamond With Simplification and Simple Comprehension applied to each of the relations $\in', <', @'$ making up V' we can ensure $V' = V$ and as $(V, \rho') \geq_u (V, \rho)$ it follows that $(V', \rho') \geq_u (V, \rho)$.

$\square$

A key tool in making this argument will be the following Translation Lemma. Intuitively, the Translation Lemma says that the way V_n, ρ_n assigns the free variables in a set theoretic formula ϕ completely determines whether $t_n(\phi)$ is true. Specifically, the truth-value of $t_n(\phi)$ which talks about how

V_n, ρ_n can be extended must agree with that of any $t_m(\phi)$ which talks about the same assignment of all the free variables in ϕ as considered within a larger hierarchy of sets V_m, ρ_n extending V_n, ρ_n.

Lemma 11.1.2 (Translation Lemma)**.** *If $v_1 \dots v_k$ are the only variables free in a set theoretic formula θ, then* $\vdash \vec{V_n} \geq \vec{V_m} \wedge \rho_n(\ulcorner v_1 \urcorner) = \rho_m(\ulcorner v_1 \urcorner) \wedge$ $...\rho_n(\ulcorner v_k \urcorner) = \rho_m(\ulcorner v_k \urcorner) \to (t_n(\theta) \leftrightarrow t_m(\theta))$.

Note, Hellman proves something analogous to this lemma in [2], assuming the axiom of inaccessibles.

Proof. I will prove this claim by induction. So suppose the claim holds for all subformula of θ and $V_n, V_m, \rho_n, , \rho_m v_1, \dots v_k$ are as in the statement of the lemma.

When θ is an atomic sentence, i.e., one of the form $x = y$ or $x \in y$, the lemma is clearly satisfied. When $\theta = \phi \vee \psi$ or $\phi \wedge \psi$ or $\neg\phi$, where the claim to be proved holds for ϕ and ψ, then this claim clearly holds for θ as well as t_n and t_m commute with truthfunctional operators.

The only non-trivial case is when $\theta = (\exists x)\phi(x)$ (as we take $\forall x$ to abbreviate $\neg\exists x\neg$). Let i be either m or n and j be the other and suppose $t_i((\exists x)\phi(x))$.

By Potentialist Translation and $\Diamond$ Ignoring we have

$$\Diamond_{\vec{V_i}, \vec{V_j}} \vec{V}_{i+1} \geq_x \vec{V_i} \wedge t_{i+1}(\phi) \tag{11.1}$$

Using 6.2.10 enter this $\Diamond_{V_i, V_j}$ context and apply Hierarchy Extending Lemma and Relabeling to establish that

$$(11.2) \qquad \Diamond_{V_{i+1},V_j} V'_{j+1} \geq V_j \wedge V'_{j+1} \geq V'_{i+1} \wedge V_{i+1} \cong_f V'_{i+1}$$

We wish to apply the inductive hypothesis to V'_{j+1} and V'_{i+1}, but first we must construct a ρ'_{i+1} such that $t'_{i+1}(\phi)$ as well as a suitable ρ'_{j+1}. So enter the above $\Diamond_{V_{i+1},V_j}$ context and import $\vec{V}_{i+1} \geq_x \vec{V}_i \wedge t_{i+1}(\phi)$. By invoking Simple Comprehension and Inner Diamond With Simplification we can assume that:

$$(\forall z, y)\rho'_{i+1}(z) = f(y) \leftrightarrow \rho'_{i+1}(z) = y$$

$$(\forall z \mid \mathbb{N}(z))\rho'_{j+1}(z) = y \leftrightarrow \begin{cases} \rho'_{i+1}(\ulcorner x \urcorner) & \text{if } z = \ulcorner x \urcorner \\ \rho_j(z) & \text{otherwise} \end{cases}$$

Clearly $(V_{i+1}, \rho_{i+1}) \cong_f (V'_{i+1}, \rho'_{i+1})$ so by Isomorphism Theorem we have that $t_{i+1}(\phi)'$. Note that as $V'_{j+1} \geq V'_{i+1}$ and $V'_{j+1} \geq V_j$ it follows that (V'_{j+1}, ρ'_{j+1}) is an interpreted initial segment.

The set of variables free in ϕ is just $x, v_1, \ldots, v_k$. By construction $\rho'_{j+1}(\ulcorner x \urcorner) = \rho'_{i+1}(\ulcorner x \urcorner)$. Corollary D.2.4 tells us that any isomorphism must be the identity on a common initial segment so for all $1 \leq l \leq k$, $\rho'_{j+1}(\ulcorner v_l \urcorner) = \rho_j(\ulcorner v_l \urcorner) = \rho_i(\ulcorner v_l \urcorner) = \rho_i(\ulcorner v_l \urcorner) = \rho'_{i+1}(\ulcorner v_l \urcorner)$. Hence, by the inductive hypothesis we can infer that $t'_{j+1}(\phi)$ and thus by Relabeling and

$\Diamond$ Introduction

$$\Diamond_{V_j} V_{j+1} \geq_x V_j \wedge t_{j+1}(\phi)$$

which is just $t_j((\exists x)\phi(x))$. As this is content restricted to V_j we can leave the diamond contexts entered for equations 11.2 and 11.1 and apply $\Diamond$ Elimination to infer $t_j((\exists x)\phi(x))$ outside of any diamond contexts completing the proof.

$\square$

Corollary 11.1.3 (Generalized Translation Lemma)**.** *If $v_1 \ldots v_k$ are the only variables free in a set theoretic formula θ, then* $\vdash V_n \geq V_0 \wedge V_m \geq V_0 \wedge \rho_n(\ulcorner v_1 \urcorner) = \rho_m(\ulcorner v_1 \urcorner) \wedge set_0(\rho_n(\ulcorner v_1 \urcorner)) ... \rho_n(\ulcorner v_k \urcorner) = \rho_m(\ulcorner v_k \urcorner) \wedge set_0(\rho_n(\ulcorner v_1 \urcorner)) \rightarrow (t_n(\theta) \leftrightarrow t_m(\theta))$.

Proof. Suppose the antecedent holds. By Pointwise Interpratation Tweaking it is possible to have $\rho_0(\ulcorner v \urcorner) = \rho_n(\ulcorner v \urcorner) = \rho_m(\ulcorner v \urcorner)$ for all variables free in ϕ. Now apply the Translation Lemma twice to infer from $t_n(\theta)$ to $t_0(\theta)$ to $t_m(\theta)$, and vice versa. Thus $t_n(\theta) \leftrightarrow t_m(\theta)$ as desired. $\square$

Note that by relabeling it trivially follows that $t^*_n(\theta)) \leftrightarrow t^*_m(\theta)$ where $t^*_n(\theta)$ is the result of replacing V_n with V^*_n in $t_n(\theta)$.

It will also sometimes be useful to deploy the following variable swap lemma.

Lemma 11.1.4. *[Variable Swap Lemma] If $\vec{V_i}$ is an interpreted initial segment with $\rho_i(v) = \rho_i(v')$, ϕ a set theoretic formula and ϕ' is the result of replacing zero or more occurrences of v in ϕ with v', provided that no bound* *variables are replaced, and all substituted occurrences of v' are free then* $t_i(\phi) \leftrightarrow t_i(\phi')$

Proof. We argue by induction on formula complexity. Suppose the assumptions in the lemma holds and the claim is provable for all subformula of ϕ. The claim is trivial if ϕ is atomic as well as if ϕ is a truthfunctional combination of subformula.

Suppose ϕ is $(\exists x)\psi(x)$. If either v or v' is x, we have $\phi = \phi'$ (and the desired result is immediate). For if $v = x$ then there are no free instances of x in $(\exists x)\psi(x)$ to replace, and if $v' = x$ then replacing any variable v in ψ with x in $(\exists x)\psi(x)$ result in capture.

Now assume that $t_i((\exists x)\psi(x))$ holds. By Potentialist Translation we have

$$\Diamond_{\vec{V}_i}[\vec{V}_{i+1} \geq_x \vec{V}_i \wedge t_{i+1}(\psi(x))] \tag{11.3}$$

Enter this $\Diamond_{\vec{V}_n}$ context. By the remarks above we can assume that v and v' are both distinct from x and $\phi' = [(\exists x)\psi'(x)]$ for some ψ' where ψ' replaces some instances of v (which are free in ψ because they are free in $(\exists x)\psi$) with instances of v' (which are free in ψ' because they are free in $(\exists x)\psi'$. As v, v' are distinct from x we have that $\rho_{i+1}(v) = \rho_{i+1}(v')$. Thus, by the inductive hypothesis we can infer $t_{i+1}(\psi'(x))$. Exiting the $\Diamond_{\vec{V}_i}$ context yields (by the definition of Potentialist Translation) $t_i((\exists x)\psi'(x)) = t_i(\phi')$. The same argument lets us derive $t_i(\phi)$ on the assumption that $t_i(\phi')$ completing the proof.

$\square$

11.2 Vindictation of FOL

Now let us return to our ultimate task: showing that every first-order logical argument in the language of set theory can be transformed into an argument in the deduction system described above, which takes us from the translation of the premises for this argument to the translation of its conclusion.

Theorem 11.2.1. *Suppose that* $\vdash_{FOL} \Phi \to \Psi$ *then* $\vdash \mathscr{V}(\vec{V_n}) \to (t_n(\Phi) \to t_n(\Psi))$

We first note this theorem has the desired claim as a corollary

Corollary 11.2.2 (Logical Closure of Translation)**.** *Suppose* $\vdash_{FOL} \Phi \to \Psi$ *then* $\vdash t(\Phi) \to t(\Psi)$

Proof. Consider any Φ, Ψ such that $\vdash_{FOL} \Phi \to \Psi$. By the theorem above, we know that $\vdash \mathscr{V}(\vec{V_n}) \to (t_n(\Phi) \to t_n(\Psi))$.

Now assume that $t(\Phi)$. By definition 10.2.5 this is just $\Box\mathscr{V}(\vec{V_0}) \to t_0(\Phi)$. From this we may infer $\mathscr{V}(\vec{V_0}) \to t_0(\Phi)$ and by using the fact that $\mathscr{V}(\vec{V_0}) \to (t_0(\Phi) \to t_0(\Psi))$ we can conclude $\mathscr{V}(\vec{V_0}) \to t_0(\Psi)$. So we have $t(\phi) \vdash \mathscr{V}(\vec{V_0}) \to t_0(\Psi)$.

Since $\mathscr{V}(\vec{V_0}) \to (t_0(\Phi) \to t_0(\Psi))$ is provable from empty premises we also have $t(\Phi) \vdash \mathscr{V}(\vec{V_0}) \to t_0(\Psi)$. So by ($\Box$ I) and the fact that $t(\Phi)$ is content restricted to the empty sentence, we can infer $t(\Phi) \vdash \Box(\mathscr{V}(\vec{V_0}) \to t_0(\Psi))$. Hence $t(\Phi) \vdash t(\Psi)$ and thus $\vdash t(\Phi) \to t(\Psi)$.

$\Box$

We will actually prove the following proposition [DISCUSS whether this should be the theorem] [discuss issue of whether this allows infinitary proofs, and the desirability of showing that they are not needed]

Proposition 11.2.3. *Given a set Γ of formulas in the language of set theory if*

$$\Gamma \vdash_{FOL} \theta$$

then

$$\mathscr{V}(V_n), t_n[\Gamma] \vdash t_n(\theta)$$

where $t_n[\Gamma]$ denotes the pointwise image of Γ under t_n.

We first note that this proposition suffices to prove the above theorem.

Proof. If $\vdash_{FOL} \Phi \rightarrow \Psi$ then $\Phi \vdash_{FOL} \Psi$ and by the above proposition we can infer $\mathscr{V}(V_n), t_n(\Phi) \vdash t_n(\Psi)$ and thus

$$\vdash \mathscr{V}(V_n) \rightarrow (t_n(\Phi) \rightarrow t_n(\Psi))$$

□

We will prove this claim by structural induction on first order proofs. But first we need a formal definition of a proof. Note that the following definition of proof makes no assumptions about the meta-language we work in other than that it has recourse to a notion of ordered tuple and is able to formally represent the various properties of formulas (so even very weak systems of arithmetic would suffice, e.g., Q, and certainly the much stronger version of number theory embedded in the system of logical possibility presented here).

Definition 11.2.4 (First Order Proof)**.** $\Gamma \vdash_{FOL} \theta$ just if there is a first order proof of θ from $\Gamma = \{\gamma_1, \ldots \gamma_m\}$ where this is inductively defined as follows (taking the various rule names are understood to refer to distinct constants[1]) and $\langle \ldots \rangle$ to denote an ordered tuple.)

(Ass) If $\theta \in \Gamma$ then $\langle Ass, \theta \rangle$ is a proof of θ from Γ.

($\wedge$I) If $\theta = \phi \wedge \psi$ and P_ϕ is a proof of ϕ from Γ and P_ψ is a proof of ψ from Γ then $\langle \wedge\mathrm{I}, P_\phi, P_\psi, \phi \wedge \psi \rangle$ is a proof of θ from Γ

($\wedge$E) If $\theta = \phi$ or $\theta = \psi$ and $P_{\phi\wedge\psi}$ is a proof of $\phi\wedge\psi$ from Γ then $\langle \wedge\mathrm{E}, P_{\phi\wedge\psi}, \theta \rangle$ is a proof of θ from Γ

($\vee$I) If $\theta = \phi\vee\psi$ and P is a proof of ϕ from Γ or ψ from Γ then $\langle \vee\mathrm{I}, P, \phi\vee\psi \rangle$ is a proof of θ from Γ

($\vee$E) If $P_{\phi\vee\psi}$ is a proof of $\phi \vee \psi$ from Γ and $P_{\phi\vdash\theta}$ is a proof of of θ from Γ, ϕ and $P_{\psi\vdash\theta}$ is a proof of of θ from Γ, ψ, then $\langle \vee\mathrm{E}, P_{\phi\vee\psi}, P_{\phi\vdash\theta}, P_{\psi\vdash\theta}, \theta \rangle$ is a proof of θ from Γ

($\rightarrow$I) If $\theta = \phi \rightarrow \psi$ and P_ψ is a proof of ψ from $\Gamma \cup \{\phi\}$ then $\langle \rightarrow \mathrm{I}, P_\psi, \phi \rightarrow \psi \rangle$ is a proof of θ from Γ

($\rightarrow$E) If $P_{\phi\rightarrow\theta}$ is a proof of $\phi \rightarrow \theta$ from Γ and P_ϕ is a proof of ϕ from Γ then $\langle \rightarrow \mathrm{E}, P_{\phi\rightarrow\theta}, P_\phi, \theta \rangle$ is a proof of θ from Γ

($\neg$I) If $\theta = \neg\phi$ and P_ψ is a proof of ψ from Γ, ϕ and $P_{\neg\psi}$ is a proof of $\neg\psi$ from Γ, ψ then $\langle \neg\mathrm{I}, P_\psi, P_{\neg\psi}, \neg\phi \rangle$ is a proof of $\neg\phi$ from Γ.

(DNE) If $P_{\neg\neg\theta}$ is a proof of $\neg\neg\theta$ from Γ then $\langle DNE, P_{\neg\neg\theta}\theta \rangle$ is a proof of θ from Γ.

[1] For instance, numbers if formalized in an arithmetic meta-language

($\forall$ I) If $\theta = (\forall v)\phi$ and P_ϕ is a proof of ϕ from some $\Gamma' \subseteq \Gamma$ with v not free in any member of Γ' then $\langle \forall \mathrm{I}, P_\phi, (\forall v)\phi \rangle$ is a proof of θ from Γ.

($\forall$ E) If $\theta = \phi(v|v')$ where v' is free for v in θ [2] and $P_{(\forall v)\phi}$ is a proof of $(\forall v)\phi$ from some Γ then $\langle \forall \mathrm{E}, P_{(\forall v)\phi}, \theta \rangle$ is a proof of θ from Γ.

$(= I)$ If $\theta = v = v$, where v is any variable then $\langle (= I), \theta \rangle$ is a proof of θ from Γ.

$(= E)$ If θ is obtained from ϕ by replacing zero or more occurrences of v_1 with v_2, provided that no bound variables are replaced, and all substituted occurrences of v_2 are free and $P_=$is a proof of $v_1 = v_2$ from Γ and P_ϕ is a proof of ϕ from Γ then $\langle (= E), P_=, P_\phi, \theta \rangle$ is a proof of θ from Γ.

($\perp$ I) If $P_{\psi \wedge \neg\psi}$ is a proof of $\psi \wedge \neg\psi$ from Γ then $\langle \perp I, P_{\psi \wedge \neg\psi}, \perp \rangle$ is a proof of $\perp$ from Γ.

($\perp$ E) If $\theta = \neg\phi$ and $P_\perp$ is a proof of $\perp$ from Γ then $\langle \perp I, P_\perp, \theta \rangle$ is a proof of θ from Γ

Note that there is no conflict between our definition of $\forall x$ as an abbreviation of $\neg\exists x \neg$ and our use of the introduction and elimination rules for $\forall$ rather than exists in proofs ($\forall E$ simply applies to statements of the form $\neg\exists v \neg\psi$).

Definition 11.2.5. If P is a first order proof then P' is a subproof of P just if either

[2]That is, if substituting v with v' does not lead to any variable which was antecedently free becoming bound. Here $\theta(v|v')$ stands for the result of substituting *all* free instances of v in θ with instances of v'.

- P has the form $\langle R, P_0, \theta\rangle$ and $P' = P_0$ or P' is a subproof P_0
- P has the form $\langle R, P_0, P_1, \theta\rangle$ and P' is P_0 or P_1 or a subproof of P_0 or P_1.

We are now in a position to prove proposition 11.2.3.

Proof. Suppose that $\vec{V}_n$ is an interpreted initial segment, $t_n(\gamma)$ holds for all $\gamma \in \Gamma$ and P is a proof of θ from Γ. Furthermore, assume, by way of induction, that the proposition holds for all P' a subproof of P. We show that $t_n(\theta)$ also holds.

Now consider the possible cases for P

$P = \langle Ass, \theta\rangle$ In this case we have $\theta \in \Gamma$ so by assumption $t_n(\theta)$ holds.

$P = \langle R, \dots\rangle$ **where** $R \in \{\wedge I, \wedge E, \vee I, \vee E, \to I, \to E, \neg I, \mathbf{DNE}\}$ This follows immediately from the fact that t_n commutes with truth functional operations and the validity of the above rules in our system for reasoning about logically possibility. For example if $\langle \wedge I, P_\phi, P_\psi, \phi \wedge \psi\rangle$ where $\theta = \phi \wedge \psi$ then $t_n(\phi \wedge \psi)$ would be $t_n(\phi) \wedge t_n(\psi)$ and by the inductive assumption applied to P_ϕ, P_ψ we know that $t_n(\phi)$ and $t_n(\psi)$ both hold yielding the desired conclusion.

$P = \langle (= I), \theta\rangle$ In this case $t_n(\theta)$ is $\rho_n(\ulcorner v \urcorner) = \rho_n(\ulcorner v \urcorner)$ which trivially follows from the assumption that V_n is an interpreted initial segment (hence ρ_n is functional with $\ulcorner v \urcorner$ in it's domain).

$P = \langle (= E), P_=, P_\phi, \theta\rangle$ If ϕ is the formula proved by P_ϕ then

By applying the inductive hypothesis to $P_=$ we may infer $t_n(v_1 = v_2)$ which by Potentialist Translation is $\rho_n(v_1) = \rho_n(v_2)$. By the inductive hypothesis applied to P_ϕ we can infer $t_n(\phi)$. As θ is obtained from ϕ by replacing zero or more occurrences of some v_1 with some v_2 in which no bound variables are replaced, and all substituted occurrences of v_2 are free the lemma lets us infer $t_n(\theta)$.

$P = \langle \forall I, P_\phi, \theta \rangle$ By definition of First Order Proof $\theta = (\forall v)\phi(v)$ for some formula ϕ and variable v and P_ϕ is a proof of ϕ from some $\Gamma' \subset \Gamma$ containing no formula γ in which v appears free.

Now, for the purposes of ($\Box$ I), suppose $\vec{V}_{n+1}$ satisfied $\vec{V}_{n+1} \geq_v \vec{V}_n$. If $\gamma \in \Gamma'$ then, as v not free in γ, by Translation Lemma we have $t_{n+1}(\gamma)$. Thus, by the inductive hypothesis applied to Γ', P_ϕ and $n+1$ we may infer $t_{n+1}(\phi)$ and thus $\vec{V}_{n+1} \geq_v \vec{V}_n \rightarrow t_{n+1}(\phi)$

As, by lemma 10.3.1, every member $t_n[\Gamma]$ is content restricted to $\vec{V}_n$ as is $\mathscr{V}(\vec{V}_n)$ and these were the only assumptions necessary to prove the above claim by ($\Box$ I) we may infer the desired conclusion

$$t_n\left((\forall v)\phi(v)\right) - \Box_{V_n} \vec{V}_{n+1} \geq_v \vec{V}_n \rightarrow t_{n+1}(\phi)$$

$P = \langle \forall \mathbf{E}, P_{\forall v \phi}, \theta \rangle$ By definition of First Order Proof θ is equal to $\phi(v|v')$ for some formula ϕ and variable v where none of the substituted instances of v' are bound.

By Pointwise Interpratation Tweaking we can infer

$$\Diamond_{\vec{V}_n} \left(\vec{V}_{n+1} \geq_v \vec{V}_n \wedge (\rho_{n+1}(\ulcorner v \urcorner) = \rho_n(\ulcorner v' \urcorner) \right)$$

Enter this $\Diamond_{\vec{V}_n}$ context. By lemma 10.3.1 we can import $t_n[\Gamma]$ and by the inductive hypothesis applied to $P_{\forall v \phi}$ we may infer

$$\Box_{\vec{V}_n} \vec{V}_{n+1} \geq_v \vec{V}_n \rightarrow t_{n+1}(\phi)$$

Application of ($\Box$ E) allows us to infer $t_{n+1}(\phi)$ and from there, as $\rho_{n+1}(\ulcorner v \urcorner) = \rho_n(\ulcorner v' \urcorner) = \rho_{n+1}(\ulcorner v' \urcorner)$ we may apply to derive $t_{n+1}(\theta)$.

As $\theta = \phi(v|v')$ if v' isn't v then v doesn't appear free in θ. If v' is v then $\rho_{n+1}(\ulcorner v' \urcorner) = \rho_n(\ulcorner v' \urcorner)$ and in either case as $\vec{V}_{n+1} \geq_v \vec{V}_n$ we have that ρ_{n+1} and ρ_n agree on all free variables in θ. Hence by Translation Lemma we can infer $t_n(\theta)$. Leaving the $\Diamond_{\vec{V}_n}$ context we have $\Diamond_{\vec{V}_n} t_n(\theta)$. Since by lemma 10.3.1 $t_n(\theta)$ is content restricted to $\vec{V}_n$ by $\Diamond$ Elimination we can conclude $t_n(\theta)$.

$\Box$