Chapter 2

Philosophical Background and the Problem to be Solved

Let me begin by introducing the potentialist approach to set theory and its motivation. I will then detail the two problems which the mathematical arguments in this monograph are intended to solve.

2.1 Motivations for Potentialism

Potentialist approaches to set theory hold that when mathematicians make claims which appear to quantify over sets, we should understand them as claims about how it is possible to extend initial segments of the hierarchy of sets. More specifically, potentialists take set theorists to be making claims about how it would be possible to have objects which satisfy the non-height related requirements of ZFC, i.e., initial segments of the sets, and how it would be possible for such objects to be extended.

Potentialist approaches to set theory provide one popular and attractive response to Burali-Forti worries about the height of the hierarchy of sets, which arise as follows.

There are well-known reasons for doubting that we have any coherent and adequate conception of absolute infinity (the supposed height of the hierarchy of sets). The concern here is not simply that it might be impossible to cash the notion of absolute infinity out in other terms. After all, every theory will have to take some notions as primitive. Rather, the worry is that it is logically impossible for any collection of objects to satisfy our intuitive notion of absolute infinity – just as Russell's paradox shows that it's logically impossible for any collection of objects to satisfy the axioms of naive set theory.

Our intuitive conception says that the hierarchy of sets goes all the way up – so no restrictive ideas of where it stops are needed to understand its behavior. However, if the sets really do go 'all the way up' in this sense, then it would seem that they should satisfy the following well-ordering principle.

> For any way some things could be well-ordered, there is an ordinal corresponding to it.

But the ordinals themselves are well ordered, and there is no ordinal corresponding to this well-ordering. Thus (it would seem), the naive well ordering principle above can't be correct.

The simplest response would seem to be to find some other restrictive characterization of the sets (in particular, some other characterization of the intended height of the hierarchy of sets).[1] However, it's not clear that any intuitive conception of the intended height of the sets remains once the paradoxical well-ordering principle above is retracted. As Wright and Shapiro put it [**?**], all our reasons for thinking that sets exist in the first place appear to suggest that, for any given height which an actual mathematical structure could have, the sets should continue up past this height.

Moreover, the sets lose a substantial aspect of their appeal as a mathematical foundation if we can't capture all talk of coherent mathematical structures within set theory – in the sense that all coherent mathematical structures have (something like) a model within the hierarchy of sets. However, it seems that this attractive principle will fail if the hierarchy of sets doesn't 'go all the way up' in the sense indicated above.

Potentialism, as developed by Putnam, Parsons and Hellman, provides a popular alternative approach to the above issue. Potentialistism holds that mathematical claims which appear to quantify over sets should (in some sense[2]) really be understood as claims about how it is logically possible to

[1] Note that the axioms of ZFC and even ZFC_2 don't suffice to categorically determine the height.

[2] Different potentialists may think of these explications of set theorists assertions in modal terms as either 'hermenutic' accounts of what contemporary set theorists already mean, or 'revolutionary' proposals for how our current mathematical concepts can be helpfully sharpened and modified in a neo-carnapian vein (to use Burgess and Rosen's terminology from []). I won't try to adjudicate this issue here, because it won't matter to the issues I will be discussing.

However, my preference is to advance potentialist paraphrases in a 'revolutionary' spirit, but only as a foundation for mathematics in the same way that Bourbaki-style reductions to set theory are currently employed as a foundations of mathematics. Thus, I'm not suggesting that set theorists should write proofs in my language, any more than Bourbaki were suggesting that number theorists should write out proofs in ZFC. Rather, I'm suggesting that we do set theory as usual, but officially note that we are (now) employing set theoretic statements as mere abbreviations for corresponding modal claims. Once we extend initial segments of the hierarchy of sets (i.e., collections of objects which satisfy our intuitive conception of the width of the hierarchy of sets but not the paradox-generating height requirement).

More specifically, potentialists take set theorists' singly-quantified existence claims, like $(\exists x)(x = x)$[3], to really be saying that that it would be logically possible for there to be an initial segment of the hierarchy of sets, V_0, containing an object x with the relevant property (here the property of being equal to itself). They take universal statements with a single quantifier like $(\forall x)(x = x)$, to really say that it is logically necessary that any object x in an initial segment of the hierarchy of sets would have the relevant property.

Potentialists handle nested quantification by using claims about how it would be logical possible for various initial segments of the hierarchy of sets to be extended. For example, they translate $(\forall x)(\exists y)(x \in y)$ as saying something like the following: 'necessarily, for any initial segment of the hierarchy of sets (call it V_1) and *any* set containing some set x, it is logically possible for there to be another initial segment (call it V_2) which extends[4] V_1 and contains a set y such that $x \in y$.

By adopting a potentialist understanding of set theory, we avoid com-

have vindicated the use of such abbreviations (by showing that all standard ZFC reasoning about set theory remains valid on the potentialist understanding as I do in this paper), set theorists can go on as usual without giving much thought to potentialism and logical possibility. However, we can pull out the fact that we are now employing set theoretic statements as abbreviations for corresponding modal claims when we need to answer philosophical questions such as the Burali-Forti problem discussed in this chapter.

[3] I mean instances of this claim as uttered in contexts where the realists would say that our quantifiers are implicitly restricted to the pure sets

[4] Meaning V_2 includes all the sets in V_1 and agrees with it regarding on the behavior of $\in$ within these sets.

mitment to arbitrary limits on the intended height of the hierarchy of sets. We also avoid the assumption that there is (or could be) any structure containing ordinals witnessing all possible well orderings. Nonetheless we make room for a sense in which the possibility of structures of arbitrary size can be relevant to the truth of set theoretic claims.

2.2 Goal 1: Blocking an Objection to Potentialism

In this paper/monograph I will attempt to address an important line of objection to potentialism. This objection concerns whether potentialists can make sense of current mainstream mathematical practice. It is not immediately obvious that the ZFC axioms (especially the axiom of replacement) remain true when understood in a potentialist manner, as statements about the possible extendability of initial segments. Thus, it is not clear that the kind of arguments mathematicians actually produce still qualify as good arguments, once we accept a potentialist understanding of what the statements in these arguments mean. Accordingly, one might fear that accepting potentialism makes current mathematical practice look unjustified and count this as a reason to reject potentialist understandings of set theory. Indeed, it's not even clear whether first order logical derivations are still valid on a potentialist understanding as potentialist paraphrases of set theorists statements change their logical structure.[5]

Geoffrey Hellman (one of the most influential potentialists in the current

[5]Obviously first order derivations as applied to statements explicitly mentioning possible extendability are valid. The question is whether first order logic when applied to statements

literature) responds to the above problems by providing a kind of 'external' justification for the use of the ZFC axioms on (his version of the) potentialist approach to set theory. Hellman's justification goes like this. Assume that actualist set theory is true and there are cofinally many inaccessible cardinals. On this assumption, we can re-interpret (Hellman's preferred version of) potentialist claims as claims about what initial segments of the true hierarchy of sets exist. Then it is a theorem that, for each first order set theory sentence ϕ, this re-interpretation of the potentialist translation of ϕ will be true iff the original sentence ϕ is true. Thus, since ZFC is presumably true of the actualist hierarchy of sets, the potentialist translation of these claims will also come out true.[6]

However, (as Hellman himself explicitly notes) this justification is not satisfactory from a potentialist point of view, because it requires that we assume the existence of an actualist hierarchy of sets. Additionally, we must also assume that this hierarchy satisfies a further (somewhat) controversial large cardinal axiom: that there are co-finally many inaccessible cardinals.

Thus, if all we have is Hellman's story, it looks like adopting a potentialist understanding of set theory makes mathematicians' current proof practices

[6]Hellman writes, "we may ask for comparisons between the [modal structuralist interpretation of set theory] and the usual fixed universe picture. It is not difficult to show that *from within the latter point of view* there is a compete agreement between the two with respect to all first order questions of ZF, decidable or not.

...the Putnam semantics [$\vDash_P$, which is interprets unbounded set theoretic claims as claims about what holds within certain initial segments of the hierarchy of sets V_k and how these V_k can be extended by other $V_{k'}$] gives answers to all mathematical questions. But does it give good ones? Yes, in this sense, it gives exactly the answers that the fixed set theoretic universe does, assuming the Axiom of Inaccessibles. That is we have a

Correctness Theorem: Let $A(\mathbf{a})$ be a sentence of $\mathscr{L}(ZF^1)$ with parameters $\mathbf{a}(= a_1...a_n)$ and suppose that $V \vDash A(\mathbf{a})$, then $\exists k$ such that V_k is a full ZFC_2 model, the a_i are in its domain, and $V \vDash_P A(\mathbf{a})$."

look unjustified. For, the potentialist is moved by Burali-Forti worries to deny the existence (and even possibility) of an actualist hierarchy of sets, such that all possible 'initial segments of a hierarchy' (in the sense relevant to potentialism) could be thought of as initial segments of this single hierarchy. Thus they should not and cannot justify their foundational principles for reasoning in set theory by appeal to such a structure.

In this monograph, I aim to solve this problem by providing a more satisfactory potentialist justification for the use of ZFC in potentialist set theory: one which (unlike Hellman's external justification) does not depend on assumptions about the acceptability of actualist set theory or any other separate mathematical structure or practice.

Rather than translating sentences of potentialist set theory back into actualist set theory and then using ZFC to prove claims, I will articulate a formal system for reasoning about logical possibility. I will then show that that the potentialist translations of each of the ZFC axioms can be derived in this formal system, and that first order inferences can be safely made.[7]

2.3 Goal 2: Justifying Replacement

Adopting my understanding of set theory also provides a new and interesting intrinsic justification for the axiom of replacement. Informally, the axiom schema of replacement says that whenever some first order formula defines a function on a set A, i.e., associates each element x of A with a unique

[7] I will also show that every first order logical deduction of a set theoretic sentence ϕ from premises Γ can be transformed into a valid deduction of $t(\phi)$ (the potentialist translation of ϕ) from $t[\Gamma]$ (the potentialist translations of all the sentences in the premise set Γ).

y, there is a set B equal to the image of A.In other words the hierarchy of sets extends far enough up that all the elements in the image of A can be collected together.

Laying aside all questions about potentialism, there is a special question about how the axiom schema of replacement is justified. If we assume an actualist understanding of the axioms for set theory, them the truth of most of the ZF axioms seem to follow directly from the cumulative hierarchy conception of the sets. However (as Boolos famously emphasized [**?**]), unlike the other axioms, replacement seems to assert something about how high the universe of sets must extend which isn't obviously a consequence of our intuitive conception of the iterative hierarchy of sets.

One might think that the axiom of replacement could be justified by appeal to the intuitive idea that the hierarchy of sets goes 'all the way up' (one can always have a long well ordering which collects together initial segments witnessing all the relevant ϕ statements). But we have already seen that this idea leads to incoherence.

Instead, the axiom of replacement is often justified 'externally' by merely appealing to the fruitfulness of the consequences we can derive from it[8] (rather than deriving it from principles which themselves are immediately compelling as is the more usual practice in mathematics). I don't deny that such external justifications can provide some support. However, it would be appealing to have a more direct argument for a claim which we use as an unargued premise when reconstructing mathematical reasoning.

Fortunately, it turns out that potentialists can provide just this kind of

[8] See Koellner on Godel on this [**?**]

justification. I will show that the potentialist translation of the replacement schema can derived from principles which seem intrinsically plausible in their own right – not just externally attractive.

Thus, the long argument presented in this monograph will show that adopting potentialist approaches to set theory can help us solve *two* antecedent problems in the philosophy of set theory. In addition to providing a principled and elegant response to the Burali-Forti paradox, accepting potentialism also allow us to provide an appealingly intrinsic justification the axiom of replacement.