Chapter 10

Potentialist Paraphrases

Let us now return to the subject of potentialist approaches to set theory. In this section, I will show how to use the notion of relitivizable logical possibility indicated above to provide attractive potentialist paraphrases for statements of (pure) set theory.

My potentialist paraphrases are inspired by Hellman's development of potentialism in *Mathematics Without Numbers.*[1]

10.1 Describing Standard-Width Initial Segments

Let me begin by introducing some definitions.

Recall the definition of well ordering from the previous section. I will

[1]I mimic Hellman's story as far as possible. However, (as noted) where Hellman translates set theory by talking about the possibility of models of ZFC_2, I do by talking about the possibility of standard-width initial segments – whatever their height I think this way of doing things is conceptually simpler and more elegant. Also (as noted above) it also lets us illuminate a way in which the axiom of replacement falls naturally out of the potentialist conception of set theory. (I also avoid Hellman's appeals to second order logic and quantifying in.)

define a formula $\mathcal{V}(set, ord, <, \in @\rangle)$ which intuitively says that some relation symbols ⟨set, ord, <, ∈ @⟩ apply like the relations '...is a set', '...is an ordinal level of the hierarchy of sets' '...is an ordinal level of the hierarchy of sets below...' '...is an element of' and '...is a set which occurs by ordinal level...' would (respectively) apply within a standard initial segment of the hierarchy of sets.[2]

Definition $\mathcal{V}(set, ord, <, \in @)$ is the conjunction of the following four requirements:

- The objects satisfying *ord* are well-ordered by <

- $(\forall x)(\forall y)[@(x, y) \to set(x) \wedge ord(y)]$

- Fatness: For each *ord* o, there are *set*s related to o by @ corresponding to all possible ways of choosing some of the *set*s which are available @ some ordinal $o' < o$ (in the sense of having exactly the chosen *set*s as elements).

[2]Thus we will, in effect, show how the notion of logical possibility can be used to specify what it takes for some relation symbols $set, \in$ etc. to apply as if to sets and ordinals within a *standard-width initial segments of the hierarchy of sets.*

This is no trivial task. Note that, for example, no sentence using only first order logical connectives can do it. All first order sentences describing the sets will have non-standard interpretations (indeed ones which are true of countable structures)

Philosophers of mathematics have traditionally tackled this problem by appealing to second order quantification to express the idea that each layer of sets must really contain objects corresponding to all possible subsets of the sets in lower layers. But we can express the same idea using the notion of relativizable logical possibility.

$$\Box_{set,ord,<,\in,@}(\forall o)[ord(o) \rightarrow$$

$$(\forall x)(P(x) \rightarrow set(x) \wedge (\exists o')(ord(o') \wedge o' < o \wedge @(x, o')))$$

$$\rightarrow$$

$$(\exists y)(set(y) \wedge @(y, o) \wedge (\forall z)(P(z) \leftrightarrow z \in y))]$$

[fix alignment]

Informally, this says that it would be impossible for a property P to apply to some sets related by @ to *ord*s below o, without there being a *set* y such that $y@o$ which contains as elements exactly the *set*s which P applies to.

- Thinness: Only those sets guaranteed by fatness exist, i.e., ,
  - Every set is available at some ordinal level.

$$(\forall x)[set(x) \rightarrow (\exists o)ord(o) \wedge @(x, o)]$$

  - All sets available at some *ord* o can *only* have *set* elements which occur at some level below as elements.

$$(\forall x)(\forall o)(@(x, o) \rightarrow (\forall z)[z \in x \rightarrow \exists o' o' < o \wedge @(z, o')])$$

  - No two distinct *set*s have exactly the same set elements.

$$(\forall x)(\forall y)[set(x) \wedge set(y) \rightarrow x = y \vee (\exists z)(set(z) \wedge \neg(z \in x \leftrightarrow z \in y)]$$

- The *ords* are disjoint from the *set*s

$$(\forall x)\neg(ord(x) \wedge set(x))$$

Note that this way of relating talk of sets to talk of ordinal levels differs slightly from the standard picture in that new sets occur at every level, whereas on the standard picture limit stages like ω just collect up the sets that occur at previous levels.

I will use $\mathcal{V}(V_i)$ to abbreviate the claim that $set_i, \in_i$ etc. satisfy the sentence $\mathcal{V}(set, ord, <, \in @\rangle)$ defined above.

10.2 Describing Standard Models of the Natural Numbers

For reasons that will become clear in the next section, it will also be useful to categorically describe the intended structure of the natural numbers, using only my relativisable $\Diamond$ operator and other first order connectives.

One can uniquely describe the intended structure of the natural numbers by combining the first 6 Peano Axioms (which can be expressed using only first order logical vocabulary) with an Axiom of Induction, which can be expressed in the language of second order logic as follows:

$$(\forall X)(([0 \in X \wedge (\forall n)(n \in X \rightarrow S(n) \in X)] \rightarrow (\forall n)(\mathbb{N}(n) \rightarrow n \in X))$$ [3]

[3] Where 0 is not officially part of our langugage, but I use claims about 0 to abbreviate corresponding claims about the the unique number that isn't a successor of anything, in the usual fashion.

Informally, this axiom says that if some property applies to 0 and to the successor of every number it applies to, then it applies to all the numbers. We can express the same idea using $\Diamond$ $\Box$ (and any one place relation P other than '$\mathbb{N}$') as follows.[4]

$$\Box_{\mathbb{N},S}[P(0) \wedge (\forall x)(\forall y)(P(x) \wedge S(x,y) \to P(y))] \to (\forall x)(\mathbb{N}(x) \to P(x))$$

This formula says that, given the facts about what is a number and a successor, i.e., about how $\mathbb{N}$ and S apply), it would be logically impossible for P to apply to 0 and to the successor of each object which it applies to without applying to all the numbers.

Call the sentence you get by replacing the axiom of induction in second order Peano Arithmetic with the above modal sentence $PA_{\Diamond}$.

10.3 The Translation

Recall that Potentialists propose to understand sentences of set theory by replacing apparent quantification over the sets with statements about how it would be possible to extend initial segments of the sets and choose elements from those initial segments, e.g., if ϕ is quantifier free then $\exists x \phi(x)$ would translate to $\Diamond[\mathcal{V}(\text{set}, \in \ ...) \wedge (\exists x)(\text{set}(x) \wedge \phi(x))]$ where this says that it would be logically possible for there to be an initial segment of the hierarchy of sets containing an object that satisfied ϕ.

To express potentialist truth conditions without quantifying in, I will require that each initial segment $\text{set}_i, \in_i, ord_i, <_i, @_i$ be paired with an as-

[4] Where $P(0)$ is shorthand for $(\exists z)(\forall w)\,(\mathbb{N}(z) \wedge \neg S(w,z) \wedge P(z))$.

sociated assignment relation R_i which assigns each of the countably many variables x_1, x_2 (where the n-th successor of 0 stands in for x_n) in the first-order language of set theory to objects within set_i. When we ask about the possibility of extending the current initial segment $(\text{set}_i, \epsilon_i)$ we can relativize further $\Box$ and $\Diamond$ to R_i requiring that an extending model must have an R_{i+1} which must agree with R_i everywhere except for on the (number representing) the variable allowed to range over set_{i+1}.

Let us say that R represents a function from the objects satisfying A to those objects satisfying B if

- R is functional, i.e., $(\forall x)(\forall y)(\forall y')(R(x,y) \wedge R(x,y') \rightarrow y = y')$
- R maps from all of A, i.e., $(\forall x)[A(x) \rightarrow (\exists y)(R(x,y))]$
- R maps to B, i.e., $(\forall x)(\forall y)(R(x,y) \rightarrow B(y))$

I will use $\mathscr{V}(V_a)$ to abbreviate the claim that set_a, ϵ_a satisfy $\mathcal{V}(\text{set}_a, \epsilon_a, ord_a, <_a, @_a)$ and R_a represents a function from the objects satisfying $\mathbb{N}$ to those satisfying set_a. More concretely, this amounts to the conjunction of the following three claims:

- $\mathcal{V}(V_a)$, i.e., set_a, ϵ_a ... behave like an initial segment of the hierarchy of sets.
- $\mathbb{N}, S$ satisfy $\text{PA}_\Diamond$ (the categorical description of the numbers above).
- R_a represents a function from the objects satisfying $\mathbb{N}$ to those satisfying set_a

Note that my only reason for using $\mathbb{N}$ is that the natural numbers (under successor) contain infinitely many definable objects, which we can use to represent variables, for example 1 represents x_1, 2 represents x_2 etc. In what follows, I will use **n**, to abbreviate the formula where **n** is replaced by a variable constrained to be the (unique) n-th successor of 0. I will use subscripts of the form $\Diamond_{V_n}$ and $\Box_{V_n}$ to abbreviate claims of the form $\Diamond_{\mathrm{set}_n,\in_n,ord_n,@_n,\leq_n,\mathbb{N},S,R_n}$ and $\Box_{\mathrm{set}_n,\in_n,ord_n,@_n,\leq_n,\mathbb{N},S,R_n}$.

I will use $V_a \geq V_b$ to abbreviate the claim that the set_a, ord_a under $\in_a, @_a, \leq_a$ extends the set_b ord_b under $\in_b, @_a, <_a$.

- $\mathcal{V}(V_a)$
- $\mathcal{V}(V_b)$
- $(\forall x)[\mathrm{set}_b(x) \to \mathrm{set}_a(x)]$
- $(\forall x)(\forall y)[\mathrm{set}_b(y) \to (x \in_b y \leftrightarrow x \in_a y)]$
- $(\forall x)[\mathrm{ord}_b(x) \to \mathrm{ord}_a(x)]$
- $(\forall x)(\forall y)[\mathrm{ord}_b(y) \to (x <_b y \leftrightarrow x <_a y)]$
- $(\forall x)(\forall y)[\mathrm{ord}_b(y) \to (x @_b y \leftrightarrow x @_a y)]$

I will use $\vec{V_a} \geq_{\mathbf{x}} \vec{V_b}$ to abbreviate the claim that $V_a \geq V_b$ and the assignment of variables R_b agrees with R_a everywhere *except on* **x**. Put more concretely, this is to say that

- $\mathbb{N}, S$ satisfy $\mathrm{PA}_\Diamond$.

- R_a represents a function from the objects satisfying $\mathbb{N}$ to those satisfying set_a
- R_b represents a function from the objects satisfying $\mathbb{N}$ to those satisfying set_b
- $(\forall n)[\mathbb{N}(n) \to n = \mathbf{x} \vee (\forall y)(R_a(n,y) \leftrightarrow R_b(n,y))]$

We can now translate the set theoretic utterance $(\exists x)(\forall y)[x = y \vee \neg y \in x]$ into a potentialist claim about how it is logically possible for $\text{set}_1, \in_1, R_1$ to be extended. First we rewrite this set theoretic statement in a regimented language with numbered variables as $(\exists x_1)(\forall x_2)[x_1 = x_2 \vee \neg x_2 \in x_1]$. Then we translate this sentence into:

$$\Diamond\, (\mathscr{V}(V_1) \wedge \Box_{V_1}[\vec{V_2} \geq_2 \vec{V_1} \to (\forall z)(\forall y)(R_2(\mathbf{1}, z) \wedge R_2(\mathbf{2}, y) \to z = y \vee \neg y \in_2 z)])$$

In words, such $\exists x_2 \forall x_1$ sentences can be understood as making a claim with, essentially, the following form. There could be a model of set theory $\text{set}_1, \in_1$ [more pedantically: a model satisfying the *width* requirements of set theory] and a relation R_1 assigning 1 (representing x_1) to an element of set_1 so that it is necessary (holding fixed $\text{set}_1, \in_1, R_1$ and the numbers) than any model of set theory $\text{set}_2, \in_2$ extending $\text{set}_1, \in_1$ and relation R_2 assigning 2 to an element of set_2 (while agreeing with R_1 about the assignment of 1) makes the interior of the above formula true when x_1, x_2 are replaced by the assignments of $1, 2$ by R_2 and $\in$ is replaced with $\in_2$.

We will make one small change to the strategy illustrated above to allow us to the quantifiers in a uniform fashion. In the above examples the first quantifier had to be treated in a special manner as (the relations abbreviated by) V_1 were not required to 'extend' any V_0. To this end, our translations will introduce a V_0 and insist that $V_1 \geq V_0$. Thus, for example, my official translation of $(\exists x)(\forall y)[x = y \vee \neg y \in x]$ is actually:

$$\Diamond [\mathscr{V}(V_0) \wedge \Diamond(\vec{V_1} \geq_2 \vec{V_0} \wedge \Box_{V_1}[\vec{V_2} \geq_2 \vec{V_1} \rightarrow (\forall z)(\forall y)(R_2(\mathbf{1}, z) \wedge R_2(\mathbf{2}, y) \rightarrow z = y \vee \neg y \in_2 z)])]$$

I will now describe recursive principles which let us translate every sentence in the first-order language of set theory into a claim about logically possible extendability.

First we define a partial paraphrase function t_n, as I do below. Intuitively, $t_n(\phi)$ transforms a set theoretic formula ϕ into the a potentialst claim about how the initial segment V_n and assignment function f_n (coded by our assignment relation R_n) can be extended so as to satisfy (a potentialist version of) ϕ – while holding fixed f_n's current assignments to all numbers representing variables which occur free in ϕ.

Definition For any number n and set theoretic formula ϕ...

- $t_n(x_i \in x_j)$ is the claim that R_n assigns the godel number of x to an object $\in_n$ the object it assigns to the godel number of y,i.e.,$(\forall z)(\forall z')[R_n(\mathbf{i}, z) \wedge R_n(\mathbf{j}, z) \rightarrow z \in_n z']$

- $t_n(x_i = x_j)$ is the claim that R_n assigns i to the same object it assigns j to i.e.,$(\forall z)(\forall z')[R_n(\mathbf{i}, z) \wedge R_n(\mathbf{j}, z') \rightarrow z = z']$
- $t_n(\neg\phi) = \neg t_n(\phi)$
- $t_n(\phi \vee \psi) = t_n(\phi) \vee t_n(\psi)$
- $t_n((\forall x)\phi(x))$ is the claim that $\Box_{V_n}[V_{n+1} \geq_{\mathbf{x}} V_n \rightarrow t_{n+1}(\phi)]$, where $\Box_{V_n}/\Diamond_{V_n}$ abbreviates a claim about what is logically necessary/possible holding fixed the facts about $\text{set}_n, \in_n, ord_n, @_n, \leq_n, \mathbb{N}, S, R_n$.
- $t_n((\exists x)\phi(x))$ is the claim that $\Diamond_{V_n}[V_{n+1} \geq_{\mathbf{x}} V_n \wedge t_{n+1}(\phi)]$

The translation of a set theoretic sentence ϕ is $t(\phi) = \Box[\mathcal{V}(V_0) \rightarrow t_0(\phi))]$.

Note that the validity of the above translation relies on the fact that for any two structures satisfying ZFC_2 one is isomorphic to an initial segment of the other. Also note that in the above definition we can replace V_j with $V_{j \bmod 2}$ without affecting the truth value of the translation. This allows us to translate sentences with arbitrarily many quantifier alternations using a fixed finite number of atomic relations.

[In what follows, I will sometimes use $\phi(f_n(x_i))$ to abbreviate claims of the form $(\exists k)R_n(\mathbf{i}, k) \wedge \phi(k)$, and f_n to abbreviate the list of relations $R_n, \mathbb{N}, S$. For ease of reading, I will also sometimes use variables $x, y, z...$ rather than $x_0, x_1, x_2....$]

10.4 Note about these translations

Lemma 10.4.1. *If $\phi, \theta_1, \ldots, \theta_n$ are formula in the language of set theory then*

1. *$t_n(\phi)$ is always content-restricted to $V_n, R_n, \mathbb{N}, S$*

2. *If ϕ is a sentence, then $t(\phi)$ is content restricted to the empty list.*

3. *For all i, j if $\mathscr{V}(V_i), t_i(\theta_1), \ldots, t_i(\theta_n) \vdash_\diamond t_i(\phi)$ then $\mathscr{V}(V_j), t_j(\theta_1), \ldots, t_j(\theta_n) \vdash_\diamond t_j(\phi)$*

Proof. Claims 1 and 2 follow immediately from the definition. Claim 3 follows by a tedious, but simple, induction on proof length, where we transform the t_i version of a proof to the t_j version by replacing every instance of a relation in V_{i+k}, f_{i+k} with the corresponding relation V_{j+k}, f_{j+k} and noting that the result is still a proof. □