Chapter 9

Stronger Infinity Lemma and the Wrapping Trick

Proposition 9.0.1. ***Infinite Well-Ordering Lemma*** *It is logically possible for there to be a non-empty well ordering with no maximal element.*

The idea behind this proof is that the *smallest* collection satisfying the conditions of the infinity axiom is well-ordered by the transitive closure of the two place relation S relating each of these objects to its successor.

So, first we note that it is logically possible for S to apply as specified by the Infinity Axiom. By Simple Comprehension it is possible for D to apply to exactly those objects which are related by successor (i.e., $(\forall x)[D(x) \leftrightarrow x \in Ext(S)]$).

By Possible Powerset, we can have a layer of classes over the objects satisfying D. By simple comprehension, it is logically possible for a predicate $SC()$ to apply to exactly those classes which are closed under S, i.e., those which contain the successor of each one of their elements. By another application of simple comprehension (and the $\Diamond$ simplification lemma), we can further specify that, simultaniously, $W, \leq$ and Z apply as follows:

- $x \leq y$ iff every successor-closed set containing x contains y , i.e., $(\forall x)(\forall y)(x \leq y \leftrightarrow (\forall k)[x \in k \wedge SC(k) \rightarrow y \in k])$.
- W applies to exactly the objects that belong to every successor-closed set containing the 0 object $(\forall x)[W(x) \leftrightarrow (\exists z)(D(z) \wedge (\forall y)\neg S(y,z) \wedge (\forall k)[SC(k) \wedge y \in k \rightarrow x \in k])]$

This completes the construction of $W, \leq$, our *intended* non-empty well ordering with no maximal element. We must now check that it really behaves as advertised.

Let me begin by proving the following lemma, which intuitively says that nothing satisfying W is equal to or less than its successor. We will use this lemma to show that $W, \leq$ satisfies the anti-reflexivity requirements. However, I will explain how to prove it now in some detail, because this lets me demonstrate the purpose and workings of a certain Wrapping Trick which will be frequently reused in the remainder of this book.

Lemma 9.0.2. $(\forall x)[W(x) \rightarrow \neg S(x) \leq x]$[1]

Proof. Consider the property of being an x such that $W(x)$ and $\neg S(x) \leq x$. By Simple Comprehension, a property G (for 'good') could apply to exactly these objects. So, by our characterization of the layer of classes over the objects satisfying D, we have added, we can infer that there's a *class* g

[1] $(\forall x)(\forall y)[W(x) \wedge S(x,y) \rightarrow \neg x \leq y]$

of such counterexamples. (Note that all objects satisfying W satisfy must satisfy D, because the class that contains all of D is successor-closed and contains 0).

If we can show that the class g of 'good' objects contains 0 and is closed under successor, then it follows (by our characterization of W) that every object satisfying W belongs to g, so no object satisfying W is $\leq$ its successor, and the lemma is true. Thus, it suffices for us to prove the following two claims.

The 0 object belongs to g, the class associated with G: Clearly, $W(0)$. We must show that $\neg S(0) \leq 0$. By our characterization of $\leq$, this means showing that there is a class which is successor-closed and contains $S(0)$, but which does not contain 0. I will argue that $\{x|W(x) \wedge \neg x = 0\}$ (i.e., the class of Ws which are not the 0 object) does the trick.

This class exists, by our characterization of the layer of classes over the objects satisfying D (and the fact that everything that satisfies W satisfies D as noted above). Obviously it does not contain 0.

It does contain $S(0)$. For we know that $S(0)$ satisfies W, because 0 satisfies W. And we know that $\neg S(0) = 0$, because 0 is not the successor of anything.

And it is successor-closed, because the objects satisfying W are closed under successor, and 0 is not the successor of anything, so $\{x|W(x)\} - 0$ must be closed under successor as well.

g is closed under successor: It suffices to prove that if x satisfies G then $S(x)$ satisfies G, i.e., $(\forall x)[W(x) \wedge \neg x \leq S(x) \rightarrow W(S(x)) \wedge \neg S(x) \leq S(S(x)))]$.

At first glance, we'd like use $\forall$-introduction to consider an arbitrary good x (i.e., an x such that $W(x) \wedge \neg x \leq S(x)$) and then show that $S(x)$ must be good as well. If we knew that there was a successor closed-class c in which S(x) is the least element, then we could show $S(S(x))$ is in this class and then construct another successor closed class c' containing S(S(x)) but not S(x), thus showing that $\neg S(S(x)) \leq S(x)$.

It might seem that we could define the needed c by comprehension as containing those elements y such that $W(y) \wedge S(x) \leq y \wedge \neg y = S(x)$.[2] However, Simple Comprehension only applies to complete sentences, not formulae with free variables.

Instead we must take a slightly more complicated approach. First we suppose, for contradiction, that the universal claim we are trying to prove is false. Then we deduce the logical possibility that some new relation $Q(\cdot)$ applies to a unique object x such that x is a counterexample to this universal claim (i.e., to a unique x, such that is good by $S(x)$ is not). In this logically possible scenario, we *can* define c as above, using Simple Comprehension over a sentence using Q rather than formula using x. Thus we can complete the above argument that $\neg S(S(x)) \leq S(x)$ (where x is taken to be the unique object satisfying Q), so $S(x)$ is good. This contradicts our initial characterization Q, yielding the conclusion that $\bot$ holds within this $\Diamond$ context. Finally we can infer from $\Diamond\bot$ to $\bot$, securing the contradiction desired.

Now let's fill in the details. The universal claim we need to prove is $(\forall x)[W(x) \wedge \neg x \leq S(x) \rightarrow W(S(x)) \wedge \neg S(x) \leq S(S(x))]$. So suppose, for

[2] i.e., it might seem that we could use simple comprehnsion to deduce that it would be that a property P could apply to exactly these objects, and then use our characterization of the layer of classes to infer that such a c exists.

contradiction, that it is false. Then there is a counterexample to it, and by Simplified Choice it is logically possible (holding fixed $S, W, \leq, class, \in, D, SC$ and any other relations we desire), that an otherwise-unused predicate Q applies to a unique object x and this object is a counterexample, i.e., $\neg[W(x) \wedge \neg x \leq S(x) \rightarrow W(S(x)) \wedge \neg S(x) \leq S(S(x)))]$.

Enter this logically possible scenario. We know that there is a unique object satisfying Q. Call it x. We also know that $\neg[W(x) \wedge \neg x \leq S(x) \rightarrow W(S(x)) \wedge \neg S(x) \leq S(S(x)))]$. .

Thus we can deduce that $W(x) \wedge \neg S(x) \leq x$. We will derive $\bot$ by proving that $W(S(x)) \wedge \neg S(S(x)) \leq S(x)$. It is easy to see that $S(x)$ exists and satisfies W.[3] As previously mentioned, we will prove that $\neg S(S(x)) \leq S(x)$ by showing that there's a successor-closed class k which contains $S(S(x))$ but not $S(x)$. By Simple Comprehension and our characterization of the classes we can show that there is a class $\{y|W(y) \wedge S(x) \leq y \wedge \neg y = S(x)\}$, because x is the unique object satisfying Q, so this class can be characterized as $\{y|W(y) \wedge (\forall x)[Q(x) \rightarrow S(x) \leq y \wedge \neg y = S(x)]\}$.[4]

This class contains $S(S(x))$ because $S(x) \leq S(S(x))$, and nothing is its

[3] $S(x)$ exists by the fact that everything satisfying W satisfies D (noted above) and the fact that everything satisfyind D has a successor. Since $W(x)$, we know that every SC class containing 0 contains x. This implies that every SC class containing 0 also contains $S(x)$, so $W(S(x))$.

[4] First we use simple comprehension to say that a predicate $P()$ could $\Diamond_{W,\leq,S,D,class,\in,Q}$ apply to exactly the objects with the property above. Then we note that this is a situation in which by our characterization of the classes must remain true (because it is content restricted to $D, class, in$). Thus we have $\Box_{D,class,\in}$(if P only applies to objects satisfying D then there is a class corresponding to the extension of P). Now we apply $\Box$E to get the truth of the conditional claim inside. We can fairly straightforwardly deduce that P only applies to objects satisfying D. So we can infer that there is a class which behaves like the extension of P, Finally, we can deduce that that there is a class containing exactly the objects which satisfy $W(y) \wedge (\forall x)[Q(x) \rightarrow S(x) \leq y \wedge \neg y = S(x)]$. Because the latter claim is content restricted to $W, \leq, S, D, class, \in, Q$, we can reason from its truth in this logically possible scenario to its truth in the scenario previously under consideration.

own successor (by antireflexivity), so $\neg S(S(x)) = S(x)$. But this class is also closed under successor. For, consider any y such that $S(x) \leq y \wedge \neg y = S(x)$. We know that $S(x) \leq S(y)$, since if every successor-closed set containing $S(x)$ contains y, then every successor closed set for $S(x)$ contains $S(y)$. But we can also show that $\neg S(y) = S(x)$, since successor is 1-1 and we know that $\neg y = x$ because $y \geq S(x)$ but (by inductive hypothesis) not $x \geq S(x)$.

Thus, there is descendent set for $S(S(x))$ which does not contain S(x), and hence $\neg S(x) \leq S(S(x))$, as desired. This contradicts our assumption that x (the unique object satisfying C) is a counterexample to the lemma. So we have $\perp$ inside this modal context.

Finally, we can pull this proof of contradiction back to our original scenario. Leaving the above $\diamond$ context, we have $\diamond_{R_1 \ldots R_n} \perp$ and hence $\perp$, which completes our proof by contradiction.

□

In the pages that follow, we will frequently use the above method of introducing a new predicate Q which applies to a single object with a given property (here it was being a counterexample to the claim that the successor of every good object is good) and then reasoning using this relation (inside the logical possibility context introduced to define it) to refer to a witness with this property. I will call this method the **Wrapping Trick**. Note that the context in which the new predicate Q is introduced will always be relativized to all other relations mentioned so far in the proof, allowing us to move needed results into this $\diamond$ context (and pull our conclusion out of it).

With the above lemma in hand, let us now turn to the mainproof. To establish that $W, \leq$ is a well ordering, we must check four things:

reflexivity: $(\forall x)(W(x) \to x \leq x)$. Consider an arbitrary x such that $W(x)$. Obviously, every set which contains x and is closed under successor contains x. So $x \leq x$.

transitivity: $(\forall x)(\forall y)(\forall z)(W(x) \wedge W(y) \wedge W(z) \wedge x \leq y \wedge y \leq z \to x \leq z)$. Consider arbitrary x, y and z such that $W(x) \wedge W(y) \wedge W(z) \wedge x \leq y \wedge y \leq z$. Clearly if every successor-closed class that contains x contains y, and every successor-closed class that contains y contains z, then every successor-closed class that contains x contains z. So $x \leq y$.

comprability. $(\forall x)(\forall y)[W(x) \wedge W(y) \to x \leq y \vee y \leq x]$.

Consider the property of of being an x such that $(\forall y)(W(y) \to x \leq y \vee y \leq x)$. Just as in the lemma above, we know that there is a class g of all the Ws which have this 'good' property[5] and I will argue that this class is successor-closed and contains 0. From this it follows that this class contains all objects satisfying W (so the comprability condition above is satisfied).

0 belongs to this class: It is immediate from our characterization of W that $(\forall y)((W(y) \to 0 \leq y)$.

If x belongs to this class then S(x) does: [We can use the Wrapping Trick (twice) to reproduce the following reasoning about a free variables x and y, using predicates Q and Q'.] Suppose, for contradiction, that some x satisfies the conditions for being 'good' but $S(x)$ does not. Then we have $(\forall y)(x \leq y \vee y \leq x)$ but also $(\exists y)\neg(S(x) \leq y \vee y \leq S(x))$. Now consider any

[5]This follows from our characterization of the layer of classes over the objects satisfying D (and the fact that every object satisfying W satisfies D noted above)

y witnessing the latter fact. We know that $x \leq y \vee y \leq x$, but:

- We cannot have $y \leq x$. For, if every successor-closed class containing y contains x, then each such class also contains $S(x)$, so $y \leq S(x)$. Contradiction.

- We cannot have $x \leq y$. For, by hypothesis, $\neg S(x) \leq y$, so there's a successor-closed class containing $S(x)$ which does not contain y. Then there's a class which contains these objects and x (by an application of simple comprehension and our characterization of the layer of classes).[6] But this is a successor-closed class containing x which does not contain y, so $\neg x \leq y$. Contradiction.

least element: $\Box_{W,\leq}[(\exists x)(K(x) \wedge W(x)) \rightarrow (\exists x')(K(x') \wedge W(x') \wedge (\forall y)[K(y) \wedge W(y) \rightarrow x' \leq y])]$

This condition asserts that (restricting our attention to the objects satisfying W) if a predicate K holds for some x satisfying W, then there is a $\leq$-least x satisfying W and K.

Suppose, for contradiction, that it were $\Diamond_{W,\leq}$ to have $(\exists x)(K(x) \wedge W(x))$ but not $(\exists x')(K(x') \wedge W(x') \wedge (\forall y)[K(y) \wedge W(y) \rightarrow x' \leq y])]$. Informally, this says that some object that satisfies W also satisfies K but there is no $\leq$-*least* such object, i.e., for every x satisfying both K and W, there is a y satisfying K and W such that $\neg x \leq y$. Call this latter claim the No Least Element Assumption, and note that, by first order logic, it can be rewritten as follows: $(\forall x')(K(x') \wedge W(x') \rightarrow (\exists y)[K(y) \wedge W(y) \wedge \neg x' \leq y]))]$.

[6] Remember that the Wrapping Trick has us consider a situation where a predicate Q applies to our putative counterexample x and Q' to some choice of y.

By Ignoring we can deduce that $\Diamond_{class,\in,W,\leq,D}((\exists x)(K(x) \wedge W(x))$ and No Least Element). Now I will consider this supposedly logically possible scenario, and derive a contradiction by showing that $\neg(\exists x)(K(x) \wedge W(x))$.

As usual, we begin by noting that there is a class g of 'good' objects satisfying W, such that nothing ≤ them satisfies K, $\{x|W(x) \wedge \forall y \neg(K(y) \wedge y \leq x)\}$.[7] I will show that this class is successor-closed and contains 0. From this it follows *all* objects satisfying W belong to this class, so nothing satisfies both K and W. This establishes the desired conclusion that $\neg(\exists x)(K(x) \wedge W(x))$.

0 belongs to this class: 0 is the only thing ≤ 0, so it suffices to check that $\neg K(0)$. But we know this is true, because the No Least Element Assumption above requires that if K(0) then there is a y such that $W(y) \wedge K(y) \wedge \neg 0 \leq y$. And there can be no such y because, by our characterization of W everything that satisfies W is ≥ 0.

This class is successor-closed: [Just as in the previous proof, we can use two nested applications of the Wrapping Trick to reconstruct the following argument, which treats x and y as free variables.] Suppose, for contradiction, that some x belongs to this class (so nothing ≤ x satisfies K) while its successor does not (so there is a y such that $y \leq S(x) \wedge K(y)$). [Consider some such x via one applicaiton of the Wrapping Trick. Then consider a witnessing y for this choiceof x via another, nested, application of the Wrapping Trick.]

First we can deduce from the fact that $y \leq S(x) \wedge K(y)$ that $y = S(x)$. We know that everything $\leq x$ doesn't satisfy K. So $\neg y \leq x$, hence there's a

[7] This follows our characterization of the layer of classes over D and the fact that all Ws are Ds (which remains true in this context because they are content-restricted to $class, \in, W, \leq, D$).

successor-closed class c containing y but not x. By simple comprehension[8], there is also another class $c' = c - \{S(x)\}$. Our original class c cannot contain any predecessor for $S(x)$, since it does not contain x and S is a 1-1 function. Thus unless $y = S(x)$, removing $S(x)$ from this original class c will leave something which contains y and is still closed under successor, but does not contain $S(x)$ (contradicting $y \leq S(x)$). Thus we have $y = S(x)$ and hence $S(x)$ satisfies K and W.

Now the No Least Element Assumption above requires that there is a z such that $\neg S(x) \leq z$ and $W(z) \wedge K(z)$. So there's a successor-closed class c containing $S(x)$ but not this z. And there is a successor-closed class $c' = c + \{x\}$ which adds x to c. We know that c' still doesn't contain z (because K(z) but not K(x) so $\neg z = x$). Thus we have $\neg x \leq z$. By the comprability property just proved above we have $z \leq x$. So we have $(\exists z)(z \leq x \wedge K(x))$, contradicting our initial choice of x.

This completes the above argument that every object satisfying W belongs to the class above, so that $\neg(\exists x)W(x) \wedge K(x)$ and we can derive $\perp$ within the $\Diamond$ context under consideration.

However, as discussed previously $\perp$ is content-restricted any list of relations, so we can infer from $\Diamond\perp$ to $\perp$. Thus we have $\neg \Diamond_{W,\leq} \neg(\exists x)[K(x) \wedge W(x) \rightarrow (\exists x')[K(x') \wedge W(x) \wedge (\forall y)(K(y) \rightarrow x \leq y)]$, as desired.

everything is either 0 or a successor: $(\forall x)[W(x) \rightarrow x = 0 \vee (\exists y)(S(y) = x)]$

[We can use the Wrapping trick above to reconstruct the following ar-

[8] And the Wrapping Trick

gument.] Suppose there is an a such that $W(a)$ but neither $a = 0$ nor $(\exists a_0)S(a_0) = a$. We can show that $\neg W(a)$ (and hence get a contradiction) by deducing the existence of a successor-closed class which contains 0 but does not contain a as follows. By our characterization of the classes over D, there is a class of all objects satisfying D. By the fact that everything in Ext(S) satisfies D and 0 has a successor, this is a successor-closed class which contains 0. There is also a class c' formed by removing only a from this class. By the assumption that a is not 0 and not the successor of anything, this class will also be successor closed and contain 0.

antisymmetry: $(\forall x)(\forall y)(x \leq y \wedge y \leq x \rightarrow x = y)$

Proof. Suppose, for contradiction, that $(\exists x)(\exists y)(W(x) \wedge W(y)x \leq y \wedge y \leq x \wedge \neg x = y)$. Using the least element condition already established, we can quickly show that there is $\leq$-least object a such that $(\exists y)(W(a) \wedge W(y) \wedge a \leq y \wedge y \leq a \wedge \neg a = y)$.[9] By the same reasoning, we can further deduce that there is an $\leq$-least b, which satisfies $W(a) \wedge W(b) \wedge a \leq b \wedge b \leq a \wedge \neg a = b$ for this $\leq$-least a.

I will argue by dilemma. First note that because $W(a)$, we know that either $a = 0$ or $(\exists a_0)(S(a_0) = a)$ by the fact just proved above.

First consider the case where $a = 0$.

I will show that $\neg b \leq 0$, contradicting our choice of b so that $b \leq a$. By simple comprehension and our characterization of the classes there is a

[9]Specifically, by simple comprehension, a property K could apply to exactly the Ws with the property above. Consider this $\Diamond$ scenario. The least element condition must remain true. So there is an $\leq$-least object a such that $(\exists y)(W(a) \wedge W(y) \wedge a \leq y \wedge y \leq a \wedge \neg a = y)$ in this scenario. But the latter claim is content restricted to $W, \leq$ so it must be true in our original scenario as well.

$c = \{x|W(x) \wedge x \geq b\}$. This class is successor-closed, since it is immediate from unpacking definitions that the successor of anything $\geq b$ is $\geq b$.

By another application of simple comprehension(as above) we know there is a class $c' = c - 0$. We can show this is also a successor-closed class, and that it containing b but not 0 (so that $\neg b \leq 0$ as desires) as follows. We know that it still contains b, because c does, 0 is the only thing that got removed from c, and $\neg a = b$ so (given our current assumption that a=0) $\neg 0 = b$. It is successor-closed because c is, and the only item which is removed (0) is not the successor of anything.

Now consider the case where $(\exists a_0)(S(a_0) = a)$. By Lemma 9.0.2 above we have $\neg S(a_0) \leq a_0$, hence $\neg a \leq a_0$. So by our characterization of a as the least thing with the property above, we know that a_0 does not have this property. So either $\neg a_0 \leq b \vee \neg b \leq a_0$ or $a_0 = b$.

- if $\neg a_0 \leq b$ then there's a successor-closed class containing a_0 which doesn't contain b, hence a successor-closed class containing a that doesn't contain b. So $\neg a_0 \leq b$. Contradiction.

- if $a_0 = b$, then since $b \leq a$ we have $a_0 \leq a$. But this is impossible by the Lemma, just mentioned.

- if $\neg b \leq a_0$, then there's a successor-closed class c containing b that doesn't contain a_0. Now it suffices to show that we can turn this into a successor-closed class containing b but not a. For, from this it follows that $\neg b \leq a$, which contradicts our choice of a.

  By our characterization of the layer of classes (and simple comprehension), we know that there is a class $c' = c - \{a\}$. The resulting class still contains b because $\neg b = a$ by the argument above. And it remains closed under successor because the single item we have removed (a) cannot be the successor of anything in c or c' (since successor is 1-1 and a is the successor of a_0 which is not in c).

□

Finally, it is easy to check that $W, \geq$ is an *infinite* well ordering in the sense characterized above: that it contains an object and contains something $\geq$-larger than every object which it contains.

infinite well ordering:

Proof. W is a well-ordering by the argument above. We know that W applies to something because clearly the 0 object satisfies W. And we can show that W contains some object strictly $\geq$-larger than every object which it contains, as follows (using the Wrapping Trick as usual).

Suppose some x satisfies W. Then $S(x)$ exists and satisfies W, as noted above.[10] Clearly every successor-closed class containing x contains $S(x)$, so $S(x) \geq x$. By the antireflexivity of S, we have $\neg S(x) = x$. So $S(x)$ is strictly $\geq$-larger than x.

□

[10]Every successor-closed class containing 0 contains x. So each of these classes also contains $S(x)$. Thus $S(x)$ also satisfies W.