THE ACCESS PROBLEM AND KNOWLEDGE OF LOGICAL POSSIBILITY

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Abstract. Accepting truth-value realism can seem to raise an explanatory problem: what can explain our true beliefs about mathematics, i.e., the match between human psychology and objective mathematical facts? A range of current truth-value realist philosophies of mathematics allow one to reduce this access problem to a problem of explaining our accuracy about which mathematical practices are coherent – in a sense which can be cashed out in terms of logical possibility. However, our ability to recognize these facts about logical possibility poses its own access problem.

I propose a solution to this residual access problem. The key idea is that accepting powerful and correct general principles for reasoning about logical possibility can be the most efficient way to predict and explain the behavior of concrete objects. Although experience with the physical world is not needed to *justify* mathematical beliefs, I will suggest that our dealings with concrete objects can *explain* how we came to employ good a priori methods of reasoning about logical possibility.

1. Introduction

It's appealing to think that there are right answers to all arithmetical questions[1] and many questions in other mathematical domains – regardless of whether our proof practices will ever let us discover these answers. However, accepting such truth-value realism raises an explanatory problem.

[1]That is, every sentence in the language of first order arithmetic, i.e., the language with function symbols + and · and constant symbol 0, is either true or false.

What can explain our true beliefs about mathematics, i.e., the match between human psychology and objective mathematical facts?

The history of past attempts to respond to this problem can make it appear that no adequate explanation of human accuracy about mathematics is conceivable[2]. Accordingly, accepting truth-value realism about mathematics can seem to require positing an extra inexplicable coincidence. This objection is sometimes called the access problem for truth-value realists about mathematics[3].

A range of current truth-value realist philosophies of mathematics (views within I will call the 'structuralist consensus') allow one to reduce the problem of explaining our accuracy about mathematics to a problem of explaining our accuracy about which mathematical practices are coherent[4] – in a sense which I will cash out using a logical possibility operator. However, this leaves us with a residual access problem concerning logical possibility.

I propose a solution to this residual access problem. The key idea is that accepting powerful and correct general principles for reasoning about logical possibility can be the most efficient way to predict and explain the behavior of concrete objects. Although experience with the physical world is not needed to *justify* mathematical beliefs, I will suggest that our dealings with

[2]For example, we can't appeal to causal contact with mathematical objects to explain the accuracy of our beliefs, because mathematical objects are causally inert. One might try to say that our use of certain mathematical proof procedures implicitly defines our mathematical terms so as to ensure that employing these proof procedures will always yield true beliefs. However, this response is difficult to reconcile with the truth-value realist intuition that all arithmetical questions must have definite right answers noted above. If there are definite right answers to all questions in the language of arithmetic and (as Gödel's First Incompleteness Theorem makes plausible [12]) there are some arithmetical questions which our proof practices do not allow us to determine the right answer to, hence ones which would not have a right answer if we equated mathematical truth with provability in some formal system.

[3]Classic formulations of the access problem, like Benacerraf's [1] have often targeted fairly traditional Platonist philosophies of mathematics. However, as we will see below, I think the best form of the access worry naturally extends to make trouble for other forms of truth-value realism as well.

[4]By this I mean semantically coherent, not just syntactically coherent.

concrete objects can *explain* how we came to employ good a priori methods of reasoning about logical possibility.

One might wonder why approaching the access problem indirectly, through logical possibility, can help. Although I don't claim that no analogous argument could be given using other primitives, I think reformulating access worries in terms of logical possibility is helpful in two major ways[5]:

First, we will see that the notion of logical possibility 'comes packaged' with particularly simple and direct expected connections to constraints on the behavior of concrete objects. In contrast, mathematical existence claims tend to be connected to facts about concrete objects via principles which are either more complicated (e.g. 'every true first order statement has a set model) or more conceptually negotiable (as witnessed by the famous incident where unexpected results in physics motivated a separation between geometry and physics rather than a revision to our theory of geometry).

Second, the apparent ontological commitments of mathematical existence claims can seem to raise an additional question about how we came to postulate the 'right' mathematical objects (as opposed to other coherent but 'wrong' objects). Approaching the access problem through logical possibility lets us bracket this question, while explaining how we came to have coherent mathematical practices.

2. Access Worries as Explanatory Demands

Now, let us consider what it means for a philosophy of mathematics to face an access problem, and what it would take to solve such a problem. Benacerraf famously introduced an access worry for platonists expressed in terms of a (now widely rejected) causal theory of reference[1].

[5]For more detail, see the contrast between my proposal and Quine's empiricism in section 6.

However most recent work, following Hartry Field[5], makes no appeal to such a constraint, and instead views access worries as arising from an unmet explanatory demand.

Various ways of cashing out this explanatory demand have been proposed. For example, Field himself says the realist must explain why, reliably, 'if mathematicians believe that P then P', and this seems hard to do in a way that is compatible with realism. Linnebo[18] says the realist must explain things like why, reliably, if mathematical sentences like "2+2=4" hadn't expressed truths, then people wouldn't have accepted them. I think we can do better[6], by construing access worries just in terms of an apparent commitment to *some* unexplained extra coincidence (beyond those required by otherwise attractive alternative approaches to the same domain)[7]. .

However, the subtle difference between these approaches will not matter for anything that follows. For, the explanatory strategy I propose will address both Field's and Linnebo's conditionals and banish any apparent commitment to an extra inexplicable coincidence.

Let me conclude this subsection by noting two important features which I take to be shared by all three formulations of the access problem above.

First, access worries turn out to be quite different from (and more troubling than) mere skepticism about mathematical objects and facts. For, access worries appear to reveal an internal tension in truth-value realists' overall theory of the world[8].

[6]Certain trivialization worries have been posed for Field and Linnebo (discussed in many places such as [3]).

[7]We have a general sense of what it would be for some regularity in the world to 'cry out for explanation.' And we tend to think that scientific or philosophical theories are, ceterus paribus, less attractive insofar as they posit regularities which cry out for explanation but include tenants which make satisfying explanation of these regularities impossible. I think that access worries are most attractively understood as a specific instance of this general epistemic fact.

[8]Specifically, they seem to reveal a tension between their beliefs about mathematics and our place in the world and their judgments about when a theory posits coincidences which cry out for explanation (and what could qualify as an adequate explanation).

Second, access worries are not (in the first instance) about justification, and may be resolved without providing any further evidence for the truth of our mathematical beliefs. Because access worries arise from the impression that no adequate explanation of human accuracy about mathematics[9] is possible, one can dissolve them merely by providing an example of such an explanation.

3. Modal Structuralism and its Access Problem

3.1. **Structuralist Consensus.** A number of current philosophies of mathematics (forming, what I call the structuralist consensus) are committed to a close relationship between coherence and mathematical facts. Such views take mathematics to be 'the science of structure', and maintain that any choice of new mathematical structures coherently extending one's current mathematical practice would succeed[10]. I have in mind views like Mary Leng's fictionalism[17], Geoffery Hellman's ante rem structuralism[14], and Quantifier Variance fueled neo-Carnapian realism about mathematical objects. [11]

As these views allow any coherent mathematical structure to be posited, they transform an explanation of our access to coherence facts into an explanation of our access to mathematical knowledge. For if we take our ability to postulate coherent, rather than incoherent, mathematical structures for granted, mathematical knowledge flows simply from application of logical

[9]That is, no such explanation which is consistent with the broad strokes of our best scientific picture of our place in the world, is possible.

[10]Of course, it is the whole of one's practice that must be coherent. So even coherent posits might be unacceptable if they are not jointly coherent with existing posits.

[11]Some of these views of the semantics of mathematical claims are hermeneutic and others are revisionary, to use Burgess and Rosen's terminology [2]. (Hermeneutic views present accounts of what we actually mean, revisionary views present accounts of what we should mean/how we should revise our practices.) I won't stress the difference here, because it won't mater to the genesis or solution of the access worries discussed here.

inferences (and as we will see those inferences are themselves expressible as claims about coherence).

Thus adopting one of these views promises to let us address access worries, if only we can solve the residual access worry about how we manage to recognize coherent mathematical posits (and coherent extensions of our current mathematical practice).

For concreteness, in the bulk of this paper I will focus on how this residual access problem arises for Modal Structuralism and can be dispelled. However, I take the solution proposed to generalize, since essentially the same kind of knowledge of coherence needs to be explained by all members of the structuralist consensus.

3.2. **Logical Possibility.** We seem to have an intuitive notion of logical possibility which applies to claims like $(\exists x)(red(x) \land round(x))$ and makes sentences like the following come out true.

- It is logically possible that $(\exists x)(red(x) \land round(x))$
- It is not logically possible that $(\exists x)(red(x) \land \neg red(x))$
- It is logically necessary that $(\forall x)(red(x)) \to \neg(\exists x)(\neg red(x))$.

Philosophers representing a range of different views of mathematics have made use of this notion[12] and are comfortable applying it to non-first order sentences[13]. Like Hartry Field, I take take it to be a primitive concept[14] not reducible to any facts about set theoretic models or possible worlds.

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[12]See the discussion of the corresponding notion of consequence in [5],[23] and [14].

[13]If you are skeptical that there is such a notion, note that it is definable in terms of the even more common notion of validity (something is logically possible iff its negation is not logically necessary iff the inference from the empty premises to its negation is not valid).

[14]At first glance, one might be tempted to identify claims about logical possibility with claims about the existence of a set theoretic model. However, philosophers like Hatry Field have convincingly argued that "We should think of the intuitive notion of validity not as literally defined by the model theoretic account, or in any other manner; rather, we should think of it as a primitive notion." in works like [7].

Modal Structuralists, like Hellman, take the true content of a pure mathematical claim to be a claim about logical possibility like $(\Diamond D) \wedge \Box(D \rightarrow \phi)$[15]. For example, if PA_2 is a sentence in second order logic which uniquely describes the the natural numbers and ϕ is a sentence in the language of arithmetic, then the modal structuralist can render the intended meaning of ϕ as $\Diamond(PA_2) \wedge \Box(PA_2 \rightarrow \phi)$[16].

If one accepts Modal Structuralism, then our knowledge of mathematics can be explained by appeal to our knowledge of logical possibility. But there's still an obvious and deeply analogous remaining worry concerning our knowledge of the (often complex and powerful) logical possibility claims need to make sense of mathematics. As facts about logical possibility are no more directly observable than facts about mathematical objects, adopting modal structuralism simply replaces one daunting access problem with another[17]. To appreciate the nature and difficulty of solving this access problem, note that recognizing facts about logical possibility requires something more than using the introduction and elimination rules for the first order logical vocabulary[18].

In what follows I will offer a *simplified story* about how creatures broadly like us could have gotten mathematical knowledge. This story will proceed

[15]Hellman treats set theory slightly differently, because of special issues about the height of the hierarchy of sets.

[16]Note that reference to properties and relations like number() or successor() can be eliminated either through second order quantification (as Hellman does [14]) or by using other relations with the same arity, as I do below.

[17]See [24] for development of the slightly different problem of accounting for accuracy about logic, e.g., how we could have come to use correct expressions like the first order logical connectives $\wedge, \vee, \neg, \exists, \forall$ with their usual introduction and elimination rule, rather than how we could come to know logical possibility facts involving these objects (which is what I'm talking about).

[18]Admittedly, by Gödel's completeness theorem, a first order logical statement requires something logically possible iff it is syntactically consistent [11] (though this fact itself can't be reliably captured in a first order fashion). However, this no longer holds if ϕ itself contains applications of $\Diamond/\Box$ or some other non-first order vocabulary (as is needed to categorically describe mathematical structures like the natural numbers).

by offering an account of how we could have come to have substantially true beliefs about logical possibility and turned this into mathematical knowledge by positing *coherent* mathematical structures.

It is important to understand that my resolution of the access problem doesn't depend on the account I give being historically accurate (and it is not intended to be). Dispelling access worries only requires dispelling the appearance that no plausible explanation of our accuracy is possible. As an analogy, consider how we might respond to an allegation by a creationist that eyes are irreducibly complex and no plausible explanation of how they evolved is possible. We might give a just-so story describing how light sensitive cells might have offered a competitive advantages to soft-bodied creatures like jellyfish without the whole apparatus of an eye and how, over time, evolutionary pressures could have favored larger amalgamations of these cells and eventually developed into a full fledged eye. Even if we doubt this is the particular sequence of events is what happened (or even discover conflicting fossil evidence) it still suffices to dispel the impression of impossibility by convincing us that some explanation consistent with all the historical facts is possible. It is in this spirit that I offer my simplified story about how creatures like us could have gotten knowledge

3.3. **Streamlining Modal Structuralism.** Existing formulations of Modal Structuralism use sentences in second order logic like PA_2 above (under the $\Diamond$ of logical possibility) to pin down the intended behavior of relevant mathematical structures. But if we use second order logic, as Hellman does, to describe the scenarios whose logical possibility we are evaluating, we would need a separate explanation of our access to facts about second order quantifiers[19]. Also, if one takes second order existence claims outside the $\Diamond$ to

[19]We would need a story about how we can be accurate about whether a concrete scenario satisfies a second order description.

generate ontological commitment, then employing second order logic threatens to re-raise the issues about object existence which focusing on logical possibility promised to let us bracket[20].

We can simplify the situation by introducing the notion of logical possibility *holding certain facts fixed* which allows us to replace second order descriptions with descriptions using only first order logic and the conditional logical possibility operator. Making this change also brings out an intrinsic unity in the subject matter to be considered. This saves us the trouble of separately explaining both our access to facts about logical possibility and second order logic.

The key to making this simplification is the fact that logical possibility claims require consideration of all possible ways a relation could apply just as second order quantification considers all possible subsets. So, for example, what distinguishes standard models for second order quantification is that they consider all second order possible subsets (not merely the definable ones). However, in order to exploit this fact, we will need a way to hold certain facts fixed when evaluating logical possibility.

To introduce the generalized notion of conditional logical possibility I have in mind, consider a sentence like, "Given what cats and baskets there are, it is logically impossible that each cat slept in a distinct basket." There's an intuitive reading on which this sentence will be true if and only if there are more cats than baskets[21]. This reading employs a notion of logical possibility *holding certain facts fixed* (in this case, structural facts about what cats and baskets there are[22]).

[20] Second order quantification is usually taken to require accepting a comprehension principle which applies to actual objects as under the $\Diamond$ of logical possibility.

[21] Admittedly, there's another reading of this sentence on which it expresses a necessary falsehood. However, this is not the reading I have in mind.

[22] Hellman's own use of logical possibility given the material facts commits him to the coherence of something very much like this notion.

Accordingly, I will use a conditional logical possibility operator $\Diamond$ which takes a sentence ϕ and a finite (potentially empty) list of relations $R_1...R_n$ and produces a sentence $\Diamond(R_1...R_n)\phi$ which says that it is logically possible for ϕ to be true, given how the relations $R_1...R_n$ apply. For ease of reading, I will sink the specification of relevant relations into the subscript as follows: $\Diamond_{R_1...R_n}\phi$.

Thus, for example, the claim, 'Given what cats and baskets there are, it is logically impossible that each cat slept in a distinct basket' becomes:

(CATS)

$$\Box_{\text{cat,basket}}\neg\Bigg((\forall x)\Big[\text{cat}(x) \to (\exists y)\,(\text{basket}(y) \wedge \text{sleptIn}(x,y))\Big]\wedge (\forall z)(\forall w)(\forall w')\Big[\text{basket}(z) \wedge \text{cat}(w) \wedge \text{cat}(w')\wedge \text{sleptIn}(w,z) \wedge \text{sleptIn}(w',z) \to w = w'\Big]\Bigg)$$

Remember that when evaluating logical possibility we consider all possibilities for the relations mentioned in the statement under consideration, whether we can describe them or not (this is the analog of requiring second order quantifiers to range over all possible collections). This will allow us to replace existing Modal Structuralist claims about the logical possibility of scenarios described in terms of second order logic with claims about how it would be logically possible for some arbitrarily chosen relations to apply.

Also note that one can perhaps get correct truth conditions by thinking of $\Diamond_{R_1...R_n}\phi$ claims as holding fixed the *particular objects* in the extension of the relations $R_1...R_n$ – and then asking *de re*, of these objects, whether one could supplement them with other objects (and choose extensions for all other relations) so as to make ϕ true. However, there seems to be a primitive notion of preserving the *structural facts* about how some relations apply, which does not depend on our understanding any such controversial *de re* claims. For example, in the case of CATS above these will be scenarios which agree with the actual world on: the number of objects satisfying cat(), the number satisfying basket() and the number of things in the extension of both cat() and basket(). However, preserving the structural facts does not require preserving facts about identity (or supposing that the relevant 'cross logical-possibility counterparthood' facts are well defined).[23].

We can also make nested claims about logical possibility. Note that in a nested claim with the form $\Diamond\Box_R\psi$, the subscript freezes the facts about how the relation R applies in the scenario being considered, which may *not* be the state of affairs in the actual world.

For example, POSSIBLY CATS (below) expresses a metaphysically necessary truth. For, whatever the actual world is like, it will always be logically possible for there to be, say, 3 cats and 2 baskets, and this scenario is one in which it is logically necessary (holding fixed what cats and baskets there are) that: if each cat slept in a basket then multiple cats slept in the same basket. So it is metaphysically necessary that POSSIBLY CATS.

[23]If, say, one cat died and an additional kitten was born, these structural facts would remain unaltered. Crudely, we might gesture at the idea of preserving the structural facts by saying that two scenarios have the same structural facts about the relations $R_1, \ldots, R_n$ if the objects satisfying (More precisely those x such that $\exists y_1, \ldots, y_k, y_{k+2}, \ldots, y_m R_i(y_1, \ldots, y_k, x, y_{k+2}, \ldots, y_m)$ for some i, k and m.) some R_i in the first scenario are 'isomorphic' (under $R_1, \ldots, R_n$) to the objects satisfying some R_i in the second scenario. Note that this is not intended to be a definition of the concept, only an attempt to point at the correct primitive notion, as the very notion of isomorphism would be defined in terms of logically possible mappings.

(POSSIBLY CATS)

$$\Diamond\Box_{\text{cat,basket}}\neg\Big((\forall x)\Big[\text{cat}(x) \rightarrow (\exists y)\,(\text{basket}(y) \wedge \text{sleptIn}(x,y))\,\Big]\wedge$$
$$(\forall z)(\forall w)(\forall w')\Big[\text{basket}(z) \wedge \text{cat}(w) \wedge \text{cat}(w')\wedge$$
$$\text{sleptIn}(w,z) \wedge \text{sleptIn}(w',z) \rightarrow w = w'\Big]\Big)$$

While I take this logical possibility to be a primitive (as Hellman does as well), in appendix A I explain how familiar set theoretic vocabulary can approximately mimic truth conditions for conditional logical possibility.

This notion of conditional logical possibility can do the same work for the modal structuralist as second order quantification. For example, using conditional logical possibility, we can express claims like the induction axiom for number theory (which is usually expressed in second order logic) as follows.

> **Induction Axiom**: if some property applies to 0 [24]and to the successor of every number it applies to, then it applies to all the numbers.

- **Induct**: 'It is logically necessary, given how number and successor apply, that if 0 is happy and the successor of every happy number is happy then every number is happy.'

By nesting logical possibility claims (asserting the logical possibility of scenarios which are themselves described in terms of logical possibility) we can write a sentence $PA_{\Diamond}$, which categorically describes the natural numbers [25] and hence ensures that for every sentence of number theory ϕ, either ϕ

[24]i.e. the unique number which is not the successor of any number

[25]Essentially, $PA_{\Diamond}$ conjoins the finitely many axioms of first order PA-Induction with the statement of induction in terms of logical possibility (Induct) above. The only other modification needed is to replace mathematical vocabulary with non-mathematical vocabulary

or $\neg\phi$ is a logically necessary consequence of $PA_\Diamond$, i.e., $\Box(PA_\Diamond \to \phi)$ or $\Box(PA_\Diamond \to \neg\phi)$. Note that while PA_2 is traditionally expressed in terms of the relations number, successor etc. we can substitute any other relations of the same arity (happy, loves etc..). As logical possibility ignores any particular features of relations (unless conditioned on) the truth value will be unaffected[26].

Using a similar method as we used with PA, we can translate the second order claim ϕ into a statement in terms of conditional logical possibility $\phi_\Diamond$ such that if $\Box(\phi \to \psi)$ then $\Box(\phi_\Diamond \to \psi_\Diamond)$[27]. As a result, any mathematical theory with a categorical description in second order logic has a categorical description in terms of conditional logical possibility[28]. Thus, using a similar strategy as above, statements about such structures (which include most definite non-set-theoretic mathematical structures) can be reduced to claims about logical possibility.

Truth conditions for mathematical investigations where no unique structure even appears to be being studied, e.g., group theory, category theory etc., can be viewed as questions about the logical possibility of structures satisfying certain criteria.[29] For instance, group theory can be viewed as asking about the logical possibility of structures satisfying the axioms for a group[30]. Thus, on this picture, an explanation of our access to facts about logical possibility would explain our access to all mathematical facts.

('number', 'successor' etc.) with non-mathematical vocabulary of the same arity, as per Putnam's suggestion above.

[26]So, for example, the sentence $\Diamond(\exists x)(\exists y)(\text{Dog}(x) \land \text{Cat}(y) \land \neg x = y)$ and the sentence $\Diamond(\exists x)(\exists y)(\text{Dog}(x) \land \text{Lemur}(y) \land \neg x = y)$ always have the same truth value.

[27]Also so that $\Diamond\phi$ implies $\Diamond\phi_\Diamond$

[28]See FILL IN for details.

[29]See [10] Hellman's discussion of this kind of math in his 1996 paper

[30]So, for instance, the claim that every group has a unique identity can be viewed as the assertion that it is logically necessary that if the relations $G, +$ satisfy the group axioms then there is a unique object satisfying G which acts as the identity under $+$.

4. Knowledge of Logical Possibility

Let us now turn to the challenge of explaining human accuracy about the logical possibility facts mentioned above. In this section, I will propose a story about how creatures like us might have developed good methods of reasoning about logical possibility.

My story will explain how creatures like us could have gotten substantial true beliefs about logical possibility by explaining how they could have gotten correct and powerful general methods of armchair reasoning about logical possibility. The key point in this explanation will be that anticipating facts about physical possibility matters to practical success and survival, and recognizing general laws of logical possibility is an efficient way to predict and explain physical possibility facts.

In particular, I will explain how people who begin with the ability to use standard first order connectives with their usual introduction and elimination rules and have some concept of (but not significant accuracy about) statements being possible or impossible with regard to the most general rules of how any objects can be related by any relations[31] could acquire powerful methods of reasoning about logical possibility sufficient to capture all of contemporary mathematics (understood in a modal structuralist vein).

As inquirers, we attempt to predict and explain the behavior of concrete objects. There are more and less economical ways of doing so[32]. When we are dealing with sufficiently diverse and plentiful collections of concrete objects, the most economical explanations for regularities may well appeal

[31]In essence, the mechanisms proposed below are supposed to explain how one could go from acceptance of very limited principles to acceptance of much more powerful principles of reasoning about logical possibility (as required to make sense of modern mathematics).

[32]Note that our compositional language (and thoughts) allows us to formulate many syntactically different descriptions of logically impossible states of affairs. Thus many plans which we can verbally represent can be discarded as physically impossible purely on the grounds that they require something logically impossible.

to a combination of general principles which constrain how any objects can be related by any relations, and specific physical or metaphysical laws whose application is restricted to certain particular kinds of objects or relations.

I will suggest that pressure to efficiently predict what is *physically possible* in situations of evolutionary interest can help explain how creatures like us could have gotten correct methods of reasoning about logical possibility[33].

4.1. **Three Mechanisms of Correction.** I will propose three key ways in which dealings with concrete objects can (directly or indirectly) help lead us towards correct methods of reasoning about logical possibility.

4.1.1. *From ϕ to $\Diamond\phi$ and $\Diamond_{R_1...R_n}\phi$ facts.* Recognizing relationships between concrete objects can push us to accept some $\Diamond\phi$ statements. Imagine that you aren't sure whether the state of affairs described by some mathematical hypothesis involving relations P, Q, and R is logically possible. If I then point out that the relations of friendship, nephew-hood and having been in military service together apply in just this way to the royal family of Sweden, this will cause you to accept that the scenario in question is logically possible.

Similarly, recognizing actual relationships between concrete objects can create systematic pressure to accept particular claims about subscripted logical possibility. Just as what is actual is logically possible, what is actual is logically possible given any facts about the actual world [34]. Thus, for example, one can go from 'every dog loves some human' to '$\Diamond_{dog}$ every dog loves some human'[35] or '$\Diamond_{human}$ every dog loves some human' or '$\Diamond_{dog,human,loves}$ every dog loves some human'. In this way recognition of actual relationships can also create pressure to accept certain $\Diamond_{R_1...R_n}\phi$ claims.

[33]Note that as discussed on pg. 7, I am not supposing that we actually used the exact notion of logical possibility.
[34]That is, for any collection of relations $R_1...R_n$ and state of affairs ϕ, if ϕ then it is logically possible that ϕ given the facts about how relations $R_1...R_n$
[35]Remember that this means considering a scenario which preserves the number of dogs in the actual world.

The advantages to be gained by recognizing useful physically possible scenarios can also create pressure to accept *general inference methods*[36] which allow one to recognize hitherto unrealized scenarios as logical possibilities. As a result, the benefits to be gained from recognizing physical possibility facts can push us towards methods of reasoning which allow us to arrive at the logical possibility of non-actual states of affairs. Similarly, it can also be useful to recognize what is possible while keeping certain relations fixed, and this can help explain our tendency to accept general inference methods which let one reliably derive true claims about subscripted logical possibility[37].

For instance, consider someone who didn't accept (even finitary) choice as a valid inference method for logical possibility. That is, they weren't willing to infer from $(\forall x)(D(x) \rightarrow (\exists y)R(x,y))$ to $\Diamond_{R,D}(\forall x)(\forall y)(F(x,y) \rightarrow R(x,y)) \wedge (\forall x)(D(x) \rightarrow (\exists! y)F(x,y))$. Such an individual might know that the enemy has divided their army up into platoons (so $D(x)$ is true just if x is a platoon in their army) and know that every platoon had at least one soldier ($R(x,y)$ holds just if y is a soldier in platoon x) but yet be unsure if it was (even logically) possible for the enemy to select a single soldier in each platoon to be the platoon leader. Failing to recognize such a possibility would be disadvantageous, and the fact that in every circumstance the question arose there was always such a choice relation would create pressure to accept such an inference procedure. Admittedly, this is a somewhat simple and contrived example but (as with mathematics) reasoning about logical possibility really comes into its own when we put multiple inferences together to reach more complex conclusions.

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[36] I take these to include things like the use of inference schemas and various ways of manipulating mental and physical pictures.
[37] that is, $\Diamond_{R_1 \ldots R_n}\phi$) facts

4.1.2. *From $\neg\phi$ facts to $\neg\Diamond\phi$ and $\neg\Diamond_{R_1\ldots R_n}\phi$ claims.* Even though the non-actual need not be non-possible, our need to elegantly explain regularities in the concrete world creates pressure to conclude certain states of affairs are logically impossible. Suppose, for example, that someone thought it was logically possible for 9 items to differ from one another in which of three properties they had, e.g., for 9 people to choose different combinations of sundae toppings from a sundae bar containing three toppings. This person would have to explain the striking law-like regularity that, regardless of the type of items and properties in question, we never wind up observing more than 8 such items. They might postulate new physical regularities to explain why apparently random processes of flipping three coins never generated the forbidden 9th possible outcome. However, this explanation (or some analogous one) would have to apply at every physical scale we can observe, from relationships between the tiniest particles to relationships between planets and stars (as well as to less concrete objects like poems and countries). A much more elegant explanation is that the unrealized outcome is logically impossible. Recognizing that the forbidden 9th outcome is forbidden in all possible domains is much more efficient than hypothesizing separate laws prohibiting it in each specific situation (and thus there is pressure to do so).

This mechanism also provides pressure to accept conditional logical possibility claims[38]. For example, if we keep noticing that when there are 4 cats and 3 baskets it is never the case that each cat slept on a different basket, the most elegant explanation for this is that it would be logically impossible for each cat to have slept on a different basket.

[38]There is pressure to recognize generalities of the form 'if ϕ then $\Box_{R,S}\phi$' because these generalizations can figure in the most elegant explanation for why it is physically impossible to bring about ϕ. Sometimes the best explanation for the fact that it is *physically* impossible to embed the objects that satisfy some relations R and S within a larger system with certain properties is that it is *logically* impossible to do so.

4.1.3. *Approaching Reflective Equilibrium.* Finally, one should note that the pressures mentioned above don't exist in isolation. Rather the resulting beliefs (and inference methods) will be further corrected by interaction with one another. If one accepts the above story about how we could have gotten some initial 'data points' about logical possibility from our knowledge of the concrete world, one can then appeal to familiar processes of reflecting on our beliefs and recognizing when they conflict or cohere with one another to explain some further improvements in our accuracy.

Once some methods of reasoning come to strike us as initially attractive via the two mechanisms above, we can arrive at new more powerful laws (just as we do in the sciences) by considering how they unify and explain these methods of reasoning. For example, in mathematics we can reliably add new axioms by choosing principles which unify and explain the mathematical beliefs which we already have[16]. As Gödel puts it, "There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems... that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory"[13]. If this is true, then it also seems plausible that the creatures in our just so story might reliably expand an initial collection of good methods of reasoning about logical possibility in the same way. Moreover, when we make incorrect generalizations these can be corrected by coming into conflict with well-entrenched and concretely motivated general principles.

Finally, remember that that the kind of elegant generalization which we see in the sciences (and which I want to invoke) goes beyond simple inferences like 'the sun rose every day for the past billion years, so it will rise tomorrow.' It can include the kind of, seemingly astonishing, leaps we see in the sciences like going from observations of points of light in the night sky to a whole model of how the planets are arranged.

4.2. **Room for A Priority and Innateness.** One might worry that the mechanisms of correction my story considers could only explain a posteriori knowledge of mathematics, while our mathematical knowledge is generally assumed to be a priori. In response to this worry, I'd like to note that correction by experience playing an important causal role in explaining our mathematical knowledge does not prevent that knowledge from seeming to us to be (or even being) a priori. Interestingly, there seem to be plenty of examples in actual history where (in what might be playfully called 'epistemic Stockholm syndrome') conscious experience forces us to believe something, and then we decide that we should have reasoned that way all along. See [27] for an intriguing real life example of this phenomenon: a case where contingent experience playing with a computer simulation using a random number generator changes people's a priori methods of reasoning about the Monty Haul problem (i.e., experience can change what methods of reasoning about probabilities people find immediately compelling *and* what methods think they should have accepted a priori). Or remember Mill's idea that [21] children learn about arithmetic by dealing with concreta and then expect these laws to hold with metaphysical necessity.

A related concern may arise for those who take our mathematical knowledge to face significant innate constraints. Someone with this perspective might find any just-so story which didn't explain how we came to have innate dispositions to reason correctly about mathematics unconvincing. In response to this concern, I'd like to note that (something like) the first two mechanisms of correction above could have been realized at an evolutionary level rendering our dispositions to accept good mathematical reasoning innate[39]. Though evolution may not care about elegance and theoretical economy in quite the sense that we do, mental resources are expensive and those methods of reasoning that could be encoded in the simplest manner and handle the most general situations would be favored.

5. Underdetermination by Evidence Worries

I will now turn to addressing a family of objections to the story I have told above. These objections arise from the following simple idea: we causally interact with relatively small collections of objects, but one needs to appeal to accuracy about the logical possibility of much larger collections (e.g., the logical possibility of objects with the structure of the sets) to explain our accuracy about mathematics. In this section I will address a number of under-determination worries which arise from this apparent gap.

5.1. **Scientific Induction Inappropriate in Mathematics?** First, one might worry that scientific-induction style generalization from cases (whether it be implemented consciously, unconsciously or evolutionary) is completely unreliable with regard to mathematics. If this were correct, it would certainly raise a problem for my proposal that dealings with small concrete collections might have pushed us to use general methods of reasoning about logical possibility which deliver correct verdicts about the kind of larger collections which are considered in pure mathematics.

However, I think we have good independent reason for rejecting the idea that generalization from cases is completely unreliable in mathematics. Mathematicians frequently use hunches developed from past experience,

[39]See Spelke's experiments with infants, [26] for an example of the kind of data which might suggest that some reasoning about what patterns of relationships between objects are (something like) logically possible are relatively innate. Further results along these lines might suggest that children have good methods of reasoning about logical possibility before they are in a position to do much personal experimentation with concrete objects, or hear good methods of reasoning advocated in the classroom.

judgments of general plausibility or theoretical attractiveness and the results of computational searches[40] to guide their research. If we want to make sense of the apparent success of this aspect of mathematical practice, we must also admit that knowledge of particular cases can be a reliable corrective to mathematical beliefs. Thus, it can't be the case that something about the nature of mathematics makes the kind of elegant generalization from cases we find in the sciences *completely unreliable* when applied to the mathematical realm. For example, belief that Fermat's last theorem was true *before* a proof found was motivated by consistent failure to find a counterexample.

5.2. **A Gap Between the Finite and the Infinite?** Next, one might worry that the story suggested above cannot explain the *degree* of mathematical knowledge which we appear to have. One might grant that the mechanisms I have suggested can explain human accuracy about what finite collections are logically possible, but hold that these mechanisms cannot explain our accuracy about logical possibility for infinite collections like the natural numbers[41].

To address this worry, I would like to note two things. First, there arguably are some countably infinite collections of physical objects which sense perception and scientific inference to the best explanation give us access to. Consider, for example, the stretches of space along the path of an arrow, or the stretches of time during which the arrow is traveling. The physical world seems to be (at least) helpfully describable in terms of some infinite

[40]Of course, they do not do this naively. If they already know that counterexamples would have to be huge they wouldn't change their judgments because no small counterexamples were found.

[41]See Frege's [8] pg. 16 for a version of this objection. He suggests that different numbers are like different geological strata and that one cannot infer facts about one from the other.

collections like the above. Plausibly this can create some pressure to acknowledge the logical possibility of certain kinds of (small) infinite systems, and to avoid unreliable reasoning about what these systems must be like[42].

Second, even if you don't accept that we have access to any infinite physical collections, reasoning about how it would be logically possible for physical objects to be supplemented with an infinite collection of abstract objects can be very useful in stating elegant laws which predict and explain the behavior of physical objects.

Consider the task of predicting what physical inscriptions of series of letters one will ever encounter. In making these predictions, it can be helpful to imagine actual physical inscriptions existing alongside a larger system of abstract objects ('strings') which witness all logically possible ways putting together finitely many letter inscriptions chosen from the relevant finite alphabet.

Even if all the inscriptions we encounter are relatively short, the most efficient way for us to recognize patterns in what inscriptions are physically possible can plausibly involve recognizing the logical possibility of strings of arbitrary finite size. This is because many 'closure principles' which smoothly (help) predict the facts about what short strings are physically possible will have the consequence that very long strings are logically possible - even strings which are too long to physically realize given the number of fundamental particles in the universe. Take, for example, the principle that for any logically possible inscription it is logically possible for there to be a 'doubled' inscription which concatenates that inscription with itself. Given the truth of principles like this, a scenario in which there are objects witnessing all logically possible choices of how to concatenate letters will be

[42]Note that I am not saying experience forces us to use a language that carves the world up into infinitely many spatio-temporal points, but only that it is sufficiently convenient and natural to do so that we can appeal to help explain our access to logical possibility.

a scenario in which there are infinitely many different abstract objects. Our methods of reasoning about logical possibility for infinite collections can be tested and corrected by the consequences they have for what this infinite collection of all strings would have to be like (and thereby, indirectly, for what physical string inscriptions are possible).

Accordingly, I think there is an adequate explanation of how dealing with finite numbers of physical objects could have lead us to recognize facts about the logical possibility of infinite collections.

5.3. **Logical Possibility and Large Collections.** A final worry concerns our access to facts about logical possibility involving larger infinite collections. Perhaps one can explain our accuracy in reasoning about countably infinite collections as above. Yet capturing intuitively correct truth conditions for statements of set theory (via the structuralist consensus) requires evaluating claims about the logical possibility of scenarios involving uncountably many objects[43]. Thus, one might worry that principles of reasoning which are shaped to elegantly predict and explain what is logically possible for finite and countably infinite collections cannot account for the degree of logical (and hence mathematical) knowledge which we actually have.

A critic might advance the following analogy: saying that elegant generalization from facts about finite and countable collections yields principles which accurately describe what is logically possible for some of the larger collections considered in pure mathematics is like saying that inference to the best explanation plus observations of birds in New Mexico explains our possession of true beliefs about birds in Canada as well. Presumably, in the ornithological case, we need to go gather more data in order to get many true beliefs about birds in Canada. But, in the mathematical case, we can't

[43]For example, such accounts would appeal to the logical possibility of satisfying the Peano Axioms which require the existence of infinitely many different numbers.

gather more data. Thus, our apparent possession of substantial true beliefs about what is logically possible for larger infinite collections remains mysterious on the story I have proposed.

I want to respond to this worry by accepting the analogy about birds above and saying that it fits the current state of human knowledge with regard to facts about the higher infinite rather well. Even in the case of birds, we can arrive at some true beliefs about birds in Canada just by inference to the best explanation from the facts about the birds in New Mexico. If we discovered tomorrow that some new island which had never yet been visited by explorers contained birds, I think we would reasonably expect many facts to carry over: any birds on that island would breathe oxygen, that they would have hollow bones etc. Our expectations about birds on this island would just be more sparse and less confident than our beliefs about birds in locations that we have observed.

But, this is just what happens with regard to our beliefs about logical possibility and large collections: as one moves from logical possibility facts concerning finite collections to those concerning countably infinite collections (like the natural numbers), and then uncountable collections (like the sets) our beliefs do get more sparse and less confident. For example, the continuum hypothesis[44] (CH) is a fairly simple statement involving sets of (relatively) small infinite size, yet it is known that both the truth and the falsity of CH are compatible with ZFC. Our beliefs about what *large* infinite collections of objects and relations are logically possible are also frequently less confident than our beliefs about what finite and countable collections of objects are logically possible. Sociologically, mathematicians are frequently

[44]The continuum hypothesis states that there are no sets whose cardinality is intermediate between the cardinality of the real numbers and that of the natural numbers. See [15] pg 176-186 for the proof that the continuum hypothesis is independent of the Zermelo-Fraenkel axioms.

much more confident in their claims about numbers, sets of numbers and sets of sets of numbers than in the distinctive claims of set theory about what much larger patterns of mathematical objects would have to be like[45].

Thus, I think this last worry points to something that is an attractive feature rather than a flaw of the account at hand: it explains why we have relatively sparse beliefs about what's logically possible with respect to large collections, and hence relatively sparse beliefs about the corresponding facts concerning higher set theory.

6. Contrast with Quine

I will conclude this paper by comparing my proposed response to access worries with the closest well-developed proposal in the literature, Quine's empiricist approach to mathematics[46]. My proposed response to access worries differs from Quine's in three major ways.

First, where Quine's proposal takes dealings with concrete objects to push us to recognize the existence of *the particular mathematical structures* which we use in the sciences, my story takes dealings with concrete objects to push us to accept correct *general inference methods* which can be used to derive the logical possibility of the structures we use in mathematics. Because my story makes the relationship between scientific and mathematical beliefs

[45]Think of choice vs. countable choice and disputes over large cardinal axioms.

[46]Modal Structualists like Hellman[14] and Shapiro[25] both gesture at the idea that something similar to Quine's approach could be used to explain knowledge of logical possibility. While their proposals only explain how we might come to accept mathematical structures indespensible to our best scientific theory, my proposal takes experience to correct our *general methods* of reasoning about logical possibility. As a result I avoid the Quinean problems about recreational mathematics noted below.

This Quinean starting point seems to have been incorporated by friends and foes of modal struturalism alike. For example in [20] Penelope Maddy argues that applications of mathematics can't explain our accuracy about set theory because everything we need to talk about has models has countable models (she is assuming physical theories are first order), as though experience had to correct our beliefs by directly forcing us to quantify over a structure rather than by motivating the acceptance of inference principles which let one describe a range of structures. I hope this and other novelties of my proposal could be accepted by them as a friendly amendment by Hellman and Shapiro.

indirect in this way it naturally avoids the 'problem of recreational mathematics' that besets Quine, i.e., the fact that we seem to know things about mathematical objects which are scientifically useless (like sets in the higher reaches of set theory).

My story also allows for the fact that (as emphasized by Michael Friedman[9]) even in cases where mathematical structures do get quantified over in physical theories, mathematicians appear to acquire significant knowledge of these mathematical objects *before* a use is found for them in physics. And it makes good sense of the apparent cavalierness of both physicists and mathematicians with regard to positing new mathematical structures [47]. For one can say that (in such cases) mathematicians and physicists are usually already convinced of general methods of reasoning which let them derive the logical possibility of suitable structures (due to prior experiences and perhaps selection on an evolutionary time scale), and, on the views in the structuralist consensus, this is enough for them to correctly[48] use such a structure.

Second, where Quine's story appeals to *continuing indispensability* mine appeals to *past usefulness*. If (as Field argues in the case of Newtonian Mechanics[6] all quantification over mathematical structures in physics is ultimately dispensable, this would be a problem for Quine's empiricism but not for my proposal. All that is necessary for my story to work is that

[47]As Justin Clark-Doane notes, physicists appear to make new mathematical postulates much more freely than they make new physical postulates – which seems odd on a Quinean picture where our acceptance of both types of objects is motivated by the same kind of inference to the best explanation. Mathematicians seem equally cavalier about positing new objects in cases where there is reason to think the relevant structure is logically possible. As Julian Cole puts it, "Reflecting on my experiences as a research mathematician, three things stand out. First, the frequency and intellectual ease with which I endorsed existential pure mathematical statements and referred to mathematical entities. Second, the freedom I felt I had to introduce a new mathematical theory whose variables ranged over any mathematical entities I wished, provided it served a legitimate mathematical purpose. And third, the authority I felt I had to engage in both types of activities. Most mathematicians will recognize these features of their everyday mathematical lives."[4]

[48]More technically (remembering the case of fictionalism) this is enough for them to be as correct as any use of mathematics is taken to be.

recognizing the logical possibility or impossibility of various claimed patterns of relationships between concrete objects was practically useful at whatever time our dispositions to reason about logical possibility were formed.

Third, while Quine says mathematical knowledge is empirical, my explanatory story is entirely compatible with mathematical knowledge being a priori[49].

7. Conclusion

In this paper, I noted that Modal Structuralism allows us to transform the classic access problem for mathematics into an access problem for knowledge of logical possibility (and many other currently popular truth value realist philosophies of mathematics seem able to do the same). I then suggested that one can solve this residual access problem by noting the role which our general methods of reasoning about logical possibility can play in our attempts to predict and explain the behavior of concrete objects.

The above argument shows that access worries can be solved for at least one form of truth-value realism. I think it is also motivates significant optimism about whether other views in the structuralist consensus (that mere accuracy about coherence is enough to guide our mathematical posits) can solve their access problems. Those views may disagree about the underlying nature of mathematical claims. For instance some views in the consensus

[49]At least on an ordinary foundationalist understanding of the a priori, which traces all a priori knowledge back to some basic principles and inferences which we can be warranted in making without justificatory appeal to anything else. If you think that *any* beliefs can qualify as basic a priori knowledge, beliefs directly produced by the application of correct general methods of reasoning about logical possibility which we find immediately compelling (and are perhaps even innately hardwired to find unquestionable) seem like an obvious candidate.

such as Quantifier Variance[50] are object-realists as well as truth-value realists while, others might spelling mathematical claims out in terms of fictions rather than logical possibility. None of this, however, prevents them from accepting that the mechanisms of correction I describe as an explanation for how creatures like us could have got reliable judgments about what is and isn't coherent.

I'd like to conclude by tentatively suggesting the following deeper picture of what is going on. Our body of mathematical theory seems to involve a mixture of insight into necessary a priori constraints on the behavior of all objects with mathematicians (oft-remarked on) apparent freedom to artistically choose which mathematical structures to talk in terms of[51].

[50]Quantifier Variantists hold that when people adopt coherent hypotheses characterizing new types of objects (like real numbers or countries), this choice can behave like an act of stipulative definition which simultaneously gives meaning to newly coined concepts by (slightly) changing the meaning of existence claims to make these statements express truths. For example, when mathematicians introduce complex numbers by adopting certain claims about their relation to the the real numbers, they shift the language we speak and change the meaning of our quantifiers to make sentences like, "there is a complex number which is the square root of -1" true. Similarly, sociologists' acceptance of ontologically inflationary conditionals like, "whenever there are people who ϕ, there is a tribe" can shift the language we speak and change the meaning of our quantifiers to ensure that this latter claim will express a truth. Even though the meaning of the (unrestricted) existential (and universal) quantifiers are changed to make the relevant stipulations express truths, the usual inferential role of the existential quantifier is preserved ("$\exists$") and the same general laws of logical possibility will constrain the behavior of objects which your new ideolect talks in terms of (thus the usual syntactic introduction and elimination rules for $\exists$ will remain valid).
This approach is ontologically realist, in the sense that it says mathematical objects literally exist and satisfies Benacerraf's famous desideratum[1] that quantification over numbers and cities look similar and hence should (ceterus paribus) be treated similarly – though not in the stronger sense of taking there to be only one supremely natural quantifier meaning which all human languages with first order logic like structure must tap into. Yet (just as much as Modal Structuralism) it allows us to reduce the access problem for mathematics to an access problem for logical possibility.
[51]As Paul Lockheart puts it in his famous essay about math education, "in mathematics... things are what you want them to be. You have endless choices; there is no reality to get in your way. On the other hand, once you have made your choices (for example I might choose to make my triangle symmetrical, or not) then your new creations do what they do, whether you like it or not. This is the amazing thing about making imaginary patterns: they talk back! The triangle takes up a certain amount of its box, and I don't have any control over what that amount is. There is a number out there, maybe it's two-thirds, maybe it isn't, but I don't get to say what it is. I have to find out what it is."[19]

Mathematics is as Quine puts it, 'black with fact and white with convention'[22] Using the notion of logical possibility lets us separate out a certain aspect of conventional choice in mathematics (which coherent package of mathematical structures to talk in terms of) leaves us with a less conventional subject matter which is a little more directly controllable and/or correctable by its applications.

Accordingly, breaking up the access problem into a question of explaining our accuracy about logical possibility and a free choice of which structures to talk in terms of is helpful in providing the kind of clear and intuitive model for how human beings could have acquired our current degree of accuracy about mathematics which is needed to truly satisfy (as opposed to merely silence) access worries.

Appendix A. A More Formal Approach to Conditional Logical Possibility

I take the notion of conditional logical possibility to be primitive and intuitive. However, one can provide approximately correct truth conditions for sentences involving nested applications of subscripted $\Box$ and $\Diamond$ operators, in terms of the more familiar language of set theory with ur-elements[52].

First let us define a formal language $\mathcal{L}$, which I will call the language of logical possibility (though this language may be not able to express all meaningful claims involving logical possibility). Fix some infinite collection of variables and a collection of relation symbols, and define $\mathcal{L}$ to be the smallest language built from these variables using these relation symbols and equality closed under applications of the normal first order connectives, quantifiers, $\Box$ and $\Diamond$ (where the latter two operators can only be applied to sentences, so there is no quantifying in).

[52] Where all the non-mathematical objects are taken to be ur-elements.

Specifically, if we ignore the possibility of sentences which demand something coherent but wouldn't have a model in the sets, (such as sentences which require the existence of proper class many objects) and take all quantifiers appearing outside a logical possibility operator to be implicitly restricted to some set sized domain of non-mathematical objects[53] we could say the following[54]:

Definition A formula ψ is true relative to a model $\mathscr{M}$ and an assignment ρ which takes the free variables in ψ to elements in the domain of $\mathscr{M}$[55] just if the following conditions obtain (note that only the last clause says something out of the ordinary):

- $\psi = R^k_n(x_1 \ldots x_k)$ and $\mathscr{M} \models R^k_n(\rho(x_1), \ldots, \rho(x_k))$.
- $\psi = x = y$ and $\rho(x) = \rho(y)$.
- $\psi = \neg\phi$ and ϕ is not true relative to $\mathscr{M}, \rho$.
- $\psi = \phi \wedge \psi$ and both ϕ and ψ are true relative to $\mathscr{M}, \rho$.
- $\psi = \phi \vee \psi$ and either ϕ or ψ are true relative to $\mathscr{M}, \rho$.
- $\psi = \exists x \phi(x)$ and there is an assignment ρ' which extends ρ by assigning a value to an additional variable v not in ϕ and $\phi[x/v]$ is true relative to $\mathscr{M}, \rho'$[56]
- $\psi = \Diamond_{R_1 \ldots R_n} \phi$ and there is another model $\mathscr{M}'$ which assigns the same tuples to the extensions of $R_1 \ldots R_n$ as $\mathscr{M}$ and $\mathscr{M}' \models \phi$.[57]

[53]Our set theoretic approximation won't be able to adequately mimic all actual objects if there are 'more' actual objects than there are sets. Note that if you are an actualist about set theory, then the machinery of conditional logical possibility lets you describe structures strictly larger than the sets, e.g. adding one layer of sets on top of V.

[54]Note that if you are a potentialist about set theory in the sense to be described below these conditions do capture correct truth conditions for logical possibility but can't be used to *define* logical possibility on pain of circularity

[55]Specifically: a partial function ρ from the collection of variables in the language of logical possibility to objects in $\mathscr{M}$, such that the domain of ρ is finite and includes (at least) all free variables in ψ

[56]As usual $\phi[x/v]$ substitutes v for x everywhere where v occurs free in ϕ, and I am taking $\forall$ to abbreviate $\neg\exists\neg$

[57]As usual I am taking $\Box$ to abbreviate $\neg\Diamond\neg$

Set Theoretic Approximation: A sentence in the language of logical possibility is true *simpliciter* iff it is true relative to a set theoretic model whose domain consists of the actual objects (which the quantifiers in our special non-mathematical object language range over) and whose extensions for atomic relations reflects the actual extensions of these relations and the empty assignment function ρ.

Note that this definition gives statements lacking any necessity operators the same truth values as they have in the actual world.

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