MATHEMATICAL KNOWLEDGE AND COMBINATORIAL POSSIBILITY: A NEW STRATEGY FOR SOLVING THE ACCESS PROBLEM

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0. Introduction

Claims like ‘there are numbers between 5 and 10’ seem to assert the existence of abstract mathematical objects like numbers and sets. But if we suppose that mathematics is really about such causally-inert abstract objects, it can seem like a mystery that creatures like us ever manage to know things about mathematics. Thus realists about mathematical objects face a kind of prima facie problem about how to account for substantial human accuracy about mathematics without positing a miracle[1]. This classic concern is typically called the access problem.

We should note that what’s at issue in the access problem is not skepticism about mathematics. All parties agree that mathematicians are (in some sense) largely right, and their claims are largely justified. The bone of contention is, rather, what we should say about the meaning of mathematical claims. What is it that mathematicians have such admirable and systematic knowledge about: abstract objects, what’s true in a given fiction, second-order logical consequence or one of the many other proposals in the literature.

To say that a theory which holds that mathematical claims describe some subject matter S faces an access problem is to argue roughly as follows.

(1) No plausible mechanism could connect human beliefs to the facts about subject matter S.

(2) Given (1) it would be miraculous if human beings had largely true beliefs about S.

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[1] See [Benacerraf, 1973] for an influential early formulation of the problem.

(3) Human beings do have largely true beliefs about whatever the true subject matter of mathematics is.

Conclusion: We have a good reason to think the subject matter of mathematics is not S.

Thus, the access problem for the realist about mathematical objects challenges her to account for the accuracy of our mathematical beliefs in light of our lack of any causal contact with mathematical objects. Importantly, we need not present any additional justification for mathematical beliefs to respond to the access problem[2]. Instead, what the realist about mathematical objects needs is some plausible story connecting human beliefs with facts about mathematical objects in such a way as to make our (purported) ability to correctly describe these objects un-miraculous[3].

In this paper I will show how we can tell such a story. First, I will show how we can characterize what it takes for mathematical objects like numbers and sets to exist in terms of a primitive modal notion which I call combinatorial possibility[4]. This reduces the problem of accounting for human access to facts about mathematical objects to a problem of accounting for human accuracy in our general methods of reasoning about combinatorial possibility[5].

Second, I will argue that there's no philosophical problem about our possession of accurate general methods of reasoning about combinatorial possibility. Facts

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[2] Indeed, many contemporary reformulations of the access problem don't even mention terms like justification or knowledge. [Field, 1980] says the Platonist's access problem is to explain the truth of the following claim, "Reliably, if mathematicians believe that ϕ then ϕ." And [Linnebo, 2006] says the Platonist must answer the following: Why is it the case that reliably: if "2+2=4" hadn't expressed a truth, we wouldn't have accepted that "2+2=4"?

[3] I should emphasize that the realist about mathematical objects does not need to *endorse* the relevant story. This story just needs to be plausible enough to defeat the impression that no naturalistic explanation of human accuracy with respect to abstract mathematical objects is possible.

[4] This proposal expands on work by [Parsons, 2005], [Putnam, 1983] and [Hellman, 1994] who have each explored various ways of giving intuitively correct truth conditions for claims about mathematical objects in terms of modal facts about what patterns of relationships are 'coherent' or mathematically possible.

[5] My story will be agnostic with respect to what particular methods of reasoning about logical possibility claims people use. This is a matter for empirical psychology, not solitary armchair investigation. What I propose to show is that, given any plausible hypothesis about what these methods of reasoning are, there's a straightforward story about how experience could correct them.

about combinatorial possibility have direct consequences for the behavior of concrete objects, by way of the fact that nothing combinatorially impossible can be actual. For example, purely a priori reasoning about how it is combinatorially possible for some n-place relations to relate a domain of objects generates the following predictions.

- It is not the case that finitely many people attended Jane's party, yet for each person who attended there was some other person who had read more books on Proust than they had.
- No finite number of physical ink marks will ever have the combination of syntactic properties required to be a proof of $0 = 1$ in Peano Arithmetic.

Now, these direct consequences of facts about combinatorial possibility for the behavior of concrete objects are not sufficient to *justify* mathematics on the basis of experience with concrete objects (as empiricists such as Mill have attempted to do). However, I will argue that our dealings with concrete objects can nonetheless play a substantial role in *explaining* the fact that we tend to deploy correct principles of reasoning about combinatorial possibility along the following lines.

As inquirers we try to predict and explain the behavior of concrete objects. There are more and less economical ways of doing so. When we are dealing with sufficiently diverse and plentiful collections of concrete objects, the most economical explanations will often appeal to a combination of general principles which are expected to constrain the behavior of all objects and relations, and specific physical or metaphysical laws whose application is restricted to certain particular kinds of objects or relations. This push to predict and explain the behavior of concrete objects by appeal to facts about combinatorial possibility can be leveraged to explain the accuracy of our mathematical intuitions with respect to the more powerful claims about combinatorial possibility which are needed to make sense of standard mathematics.

In §1 of this paper I will pin down the relevant notion of good principles for reasoning about combinatorial possibility. In §2 I will present a certain characterization of 'what it takes for there to be a number', such that all further mathematical truths about the numbers are combinatorially necessary consequences of this characterization[6]. In §3 I will discuss some mechanisms by which dealings with concrete objects could have given us good general methods of reasoning about combinatorial possibility. In §4 I will address worries about whether the above mechanisms could plausibly have given us accurate intuitions about combinatorial possibility with sufficient power to reconstruct standard mathematics.

1. Combinatorial Possibility

1.1. Introducing the Notion of Combinatorial Possibility.

Let us begin with the notion of combinatorial possibility[7]. To motivate this notion, consider the sentence below:

> **SNOBS:** There are finitely many people at the party, but for each partygoer there is some other partygoer that owns more books on Proust than they do.

One can know that SNOBS is false without doing any psychology or investigating the metaphysics of owning or the nature of being a book. SNOBS simultaneously demands that both that there are only finitely many partygoers, and that for each partygoer, A, there is some other partygoer, B, such that B bears the 'owns' relation to more items in the extension of 'book on Proust' than A[8]. This demand seems to violate very general constraints on how *any* objects can be related by *any* relations.

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[6] For reasons of space, I will focus in this paper on showing how to account for human knowledge of facts about the numbers, but see the the appendix to the longer document on my website www.seberry.org for discussion of how the same story can be extended to account for knowledge of the sets and other mathematical objects.

[7] In the literature the labels mathematical possibility [Parsons, 2007] and coherence [Shapiro, 1997] have been used to describe what I think is essentially a similar idea. So as not to make any controversial interpretive assumption I will use the distinct label combinatorial possibility for my notion.

[8] To avoid undesirable complexity we don't provide the full translation of the concepts finitely and more in purely combinatorial vocabulary at this point but note that such a translation does exist.

I take examples like this to suggest that we have a grip on some sense in which the mere (broadly) logical structure of what a sentence demands can ensure that the sentence is false. We have some notion of how it would be in principle possible for any n-place relations to apply to any objects, and we are able to discern that none of these in principle possible choices for how relations like ‘owns’ and ‘is a book on Proust’ could apply would suffice to make SNOBS true. Thus, even without considering anything particular about the relation of owning or the property of being a book on Proust, we can see that SNOBS must be false.

In the remainder of this section I want to flesh out the idea that there are primitive modal facts about combinatorial possibility in two ways. First, I will show how the notion of combinatorial possibility constitutes a natural extension of the more familiar notion of logical possibility in propositional logic and in so doing help pin down exactly what I mean by combinatorial possibility. Second, I will explain one way in which a concept can be primitive and argue that combinatorial possibility is primitive in that sense.

1.2. **How Combinatorial Possibility Extends Notions from Propositional Logic.** One can view the notion of combinatorial possibility as a kind of natural generalization of the more familiar notions of logical possibility, tautology and contradiction which occur in propositional logic. Think about what happens when we say that a sentence like, ‘If it is not raining then it is not the case that both it is raining and it is snowing’ expresses a tautology. First, we think of this sentence as being built up out of atomic propositions and logical connectives in some particular way. Thus, for example, we think of the claim mentioned above as having the form $\neg r \supset \neg (r \wedge s)$.

Second, given this choice of a formalization, we consider a range of ways it would be possible to assign either truth or falsehood to each of the different atomic propositions which figure in our formalization. We might write down a truth table listing all these different possible assignments. Note that when doing this we consider the same full range of ‘combinatorially’ possible assignments of truth-values to atomic sentences when considering any two atomic propositions r and s. We don't worry about whether it would be metaphysically or physically possible for 'it is raining' to be true while 'it is snowing' is false before writing down a line on our truth table corresponding to this possibility.

Third, we evaluate whether the whole sentence expresses a tautology, a contradiction, etc. by considering how whatever logical connectives are mentioned in the formalization would apply to determine a truth value for the whole sentence given each one of the possible assignments specified above. We say a proposition is a tautology if it comes out true for every such assignment, a contradiction if it is true for none, and a (propositional logical) contingency if it is true for some.

Now the modal notion which I am calling combinatorial possibility arises from generalizing this idea to the case of predicate logic in the following way. As in the case of propositional logical possibility, combinatorially possibility or impossibility only applies to a proposition to the extent that we can think of that proposition as having a certain logical form. However, the relevant logical form is now one that demands a certain pattern in *how n-place relations relate objects*, rather than a pattern of *truth and falsehood for atomic sentences*. Instead of formalizing sentences in terms of atomic propositions which are related via boolean connectives like $\wedge, \vee$ and $\neg$, we now formalize sentences in terms of atomic relations and the classical connectives of first order logic together with any other 'combinatorial' vocabulary which help systematically determine the truth value for a sentence based only on facts about the size of a domain and the extensions of predicate symbols within that domain.

Given this formalization, we again turn to consider a range of in principle possible states of affairs with regard to the basic components that figure in our formalization. Here however, instead of determining the truth or falsity of a sentence based on an assignment of truth or falsity to atomic sentences, our combinatorial vocabulary now determines the truth or falsity of a sentence based on something more complicated: a choice of domain and extensions for predicates within this domain.

Thus, rather than considering a space of all 'in principle possible' ways of assigning truth values to different atomic sentences, we must now think of a space of all in principle possible ways of choosing how many objects the world is to contain and which of these objects fall in the extension of the predicates in the formalized sentence.

Now what do I mean by the space of all in principle possible choices of domain and extension? In the case of propositional logic above we ignored all physical and metaphysical relations that might apply to the propositions represented by the atomic propositions and simply considered all ways to assign truth values to those propositions. I think the natural analog of this idea to the richer logical structure we use to analyze combinatorial possibility is to consider the full range of ways it would be possible to choose a domain and relations on that domain, ignoring:

- any limits on the size of the domains
- any particular metaphysical or physical facts about the nature of the relations involved which constrain how they apply (e.g., if we formalize a sentence in such a way as to make raven(x) and vegetable(x) separate predicates, we will consider models that assign some objects to the extension of both predicates, even though it is presumably metaphysically impossible for any one object to be both.)

Finally, once we have this range of, as I shall put it, all combinatorially possible *choices of domain and extension for predicates* in mind, we can use it to classify *sentences* as possible or impossible with respect to the pattern of n-place relations applying to objects which they demand. This classification proceeds in much the same way as we the classification of sentences as tautologous, contradictory, etc. with respect to the pattern of truth conditions for atomic sentences that they demand. I will say that a (formalized) sentence requires something that is **combinatorially possible** iff the pattern of relationships which that sentence demands from the world is satisfiable by some combinatorially possible choice of domain and extension for its predicates. Conversely a (formalized) sentence requires something that is **combinatorially necessary** if and only if every combinatorially possible choice of domain and extension for its predicates makes that sentence false. Thus the notion of combinatorial possibility is a predicate logical analog of the notion of logical contingency in propositional logic and the notion of combinatorial necessity is a predicate logical analog of the notion of tautology in propositional logic.

We should note that the resulting notion of combinatorial possibility differs from more familiar reductive conceptions of logical truth in some important ways. Reinterpretation based conceptions of logical truth, following Bolzano, say that the logical truths are those sentences which are content neutral in the sense that they remain true under all ways of reinterpreting the sentence by replacing one n-place relation with another and (perhaps) restricting the domain of the quantifiers [9]. My view differs from these insofar as it doesn't require that witnessing models be made by choosing some objects from the actual world as the domain of the model. This has the attractive consequence that intuitively contingent facts about the size of the universe have no effect on facts about what's combinatorially possible.

Unlike the notion of propositional logical possibility the notion of combinatorial possibility admits a natural extension to sentences already making claims about combinatorial possibility. To do this we must introduce the notion of a sentence being combinatorially possible holding fixed certain relations. So, for example, we might say things like, "Given what kittens and baskets there are, it is combinatorially impossible that each kitten to have slept in a distinct basket" which will be true if and only if there are at least as many baskets as kittens in the actual world. See Appendix A for more details.

1.3. **Primitiveness.** Now let me turn to the sense in which I want to suggest that facts about combinatorial possibility are primitive. The notion of combinatorial possibility I have indicated above is closely related to standard definitions of first order consistency. Indeed, were we in a position to take facts about the sets for granted, we could define $\Diamond$ in terms of the existence of set theoretic models for

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[9] [Etchemendy, 1990]

a sentence which are standard with regard to their interpretation of various expressions[10]. However, I want to suggest that facts combinatorial possibility require no grounding in the existence of objects of any kind, be they physical objects or abstract set theoretic models. In this regard, the view I am suggesting is analogous to a standard position on metaphysical possibility: taking metaphysical possibility to be primitive, as opposed to analyzing it in terms of the actual existence of Lewissian possible worlds [Lewis, 2001]. Just as a realist about possible worlds can give accurate truth conditions for claims about metaphysical possibility in terms of possible worlds, a realist about mathematical objects can give accurate truth conditions for the claims about combinatorial possibility in terms of sets. But (presumably) we need not think about the metaphysical possibility of there being, say, a pink elephant as being made true by the actual existence of a Lewissian possible world or any other object. Similarly, I want to suggest that there are primitive, fundamental modal facts about what patterns of relationships between objects are combinatorially possible.

To motivate this idea, recall our comparison between facts about combinatorial possibility and facts about propositional logical possibility. We can determine whether or not a sentence is a propositional logical possibility by drawing up a truth table, but presumably the fact that a sentence is a propositional logical possibility does not require grounding in the existence of a truth table. If you buy the analogy between combinatorial possibility and propositional logical possibility suggested above, then it seems natural to suppose that facts about combinatorial possibility require no more grounding in the existence of some witnessing mathematical objects than facts about propositional logical possibility require grounding in the existence of truth tables.

I think another substantial motivation for taking facts about combinatorial possibility to be freestanding in the way I have suggested is that it allows for an attractive solution to the access problem. But there, of course, the proof is in the

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[10] Note that this would only capture the correct truth conditions in the intended model of set theory.

pudding; I invite you to read the rest of this paper and make up your own mind on how attractive my proposed solution is.

2. Mathematical Objects and Combinatorial Possibility

Now let us turn to the topic of mathematical objects.

When one considers the variety of different kinds of objects that apparently exist, it seems plausible that the existence of a table can require something very different from the world than the existence of a hole. Similarly, the existence of a hole requires something very different from the world from that of an electron, a wave, a person, a whirlpool or a company. Along these lines, I want to propose that what it takes from the world for there to be a number is for certain things to be combinatorially possible - just as you might think what it takes for there to be a hole in the sofa is for the material making up the sofa to be arranged in a certain way.

This modal approach to facts about the numbers promises to reduce the problem of access to number to a problem of access to facts about combinatorial possibility, as follows.

David Lewis and others have argued that holes are best thought of as objects distinct from the physical matter which hosts or fills a hole[11]. If we accepted this view, would there be an access problem regarding our knowledge of how holes are distributed? Plausibly there is no such access problem, because of the tight connection between facts about the distribution of matter and facts about what holes exist. All it takes for there to be a hole is for matter to be distributed in a certain way. And, in view of this close, loosely-speaking definitional, relationship between facts about matter and facts about holes, it is no mystery that creatures who were evolved to be able to track the location of matter could also (if they happened to have the concept of holes) track the locations of holes as well.

I propose that there's a similar tight connection between the existence of mathematical objects and facts about combinatorial possibility. What it takes for a

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[11] See, for example, [Lewis and Lewis, 1970].

number with a certain property to exist[12] is for certain facts about combinatorial possibility to obtain. I will argue below that it is possible to give an implicit definition of numbers purely in terms of combinatorially possibility which specifies everything about what natural numbers exist and how they are related to one another. In view of this connection I propose that, just as if we have an account of human access to facts about matter we can use this to explain how we have access to facts about holes, if we have an account of human access to facts about combinatorial possibility we can thereby explain how we have access to facts about the numbers and the sets[13].

More specifically, I propose that the characterization H, below, implicitly defines what it takes for there to be a number with some property $\phi(x)$, just as bridge laws connecting facts about the distribution of matter to facts about holes might implicitly define what it takes for there to be a hole. In particular I suggest that there is a number satisfying $\phi(x)$ if and only if $\exists(x)(\mathbb{N}(x) \wedge \phi(x))$ is combinatorially necessitated by H[14]. Informally what H asserts is that there is a collection of objects satisfying $\mathbb{N}(x)$ and a relation $\leq$ on those objects such that: H:

(1) The numbers are well-ordered by the $\leq$ relation.

(2) There are *enough* numbers that it would be combinatorially impossible for a finite well-ordering of objects not to be isomorpic to some initial segment of the numbers.

(3) The numbers are *as few as can be* given that they satisfy (1) and (2)

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[12] Including properties such as being the 5th successor of 0, i.e., being 5.

[13] See my paper, ‘Stipulative Knowledge of Existence Facts and Eklund’s Criticisms of Quantifier Variance’ available at www.seberry.org for my preferred account of our knowledge of the relationship between facts about holes the distibution of matter, facts about numbers and combinatorial possibility etc.

[14] That is, it depends on whether $\Box\,(H \supset \exists(x)(\mathbb{N}(x) \wedge \phi(x)))$

We can then define expressions like 0 and successor in terms of $\leq$ and number in the natural way[15]. We can further show that H suffices to determine intuitively correct truth conditions for all statements in the language of number theory[16]. Using an argument similar to that which is used to show the categoricity of second order arithemtic we can show that for each such statement ϕ it is either combinatorially necessary that $H \supset \phi$ or combinatorially necessary that $H \supset \neg\phi$. (See Appendix B for details on how to express the clauses in H in terms of combinatorial possibility, and how the above-mentioned proof works.)

This power of my proposed characterization is important because it lets us ground facts about number existence in facts about combinatorial possibility without giving up the intuitive idea that all questions in the language of number theory have definite right answers. Many attempts to understand mathematical truths as analytic consequences of some kind of implicit definition of the numbers have frequently failed at this point. For example, we can't understand mathematicians' claim that ϕ as saying that ϕ is derivable from a characterization like H in standard first-order logic without sacrificing the above intuition. This is because, by the First Incompleteness Theorem any algorithm that lists only truths in the language of arithmetic will leave something out, failing to list either ϕ or $\neg\phi$ for some ϕ in the language of arithmetic. Thus neither first-order logic nor any notion of consequence $\vdash$ which allows for a complete formal proof procedure can make all truths of arithmetic out to be consequences of some finite or recursively enumerable characterization of the numbers.

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[15] x is the successor of y if and only if

$$y \leq x \wedge x \neq y \wedge (\forall z)\,[z \leq x \supset (z \leq y \vee z = x)]$$

We then define 0 to be the unique number that it is not the successor of any other number. We can then define + in terms of a nested statement about combinatorial possibility by saying that $z = x + y$ if it is possible to pair up the numbers less than z with the disjoint union of those less than x and less than y.

[16] By sentences in the language of number theory I mean, in essence, sentences that only talk about how the numbers are related to one another under 0, successor and plus. More formally, I mean sentences built up out of atomic formulae containing variables, the relation le, the formal paraphrases of 0, successor and plus, equality and the sentential connectives, together with quantifiers of the form $(\forall x)(\mathbb{N}(x) \supset \ldots)$ and $(\exists x)(\mathbb{N}(x) \wedge \ldots)$.

In contrast, like characterizations in terms of second order logic, we *can* give a description of the numbers H which has all truths in the language of number theory as necessary consequences. This is possible because unlike first-order logical consequence, the notion of combinatorially necessary consequence is not characterized by any exhaustive algorithm which derives all consequences of the relevant kind. It thereby gives us a grip on a notion of 'consequence' which can outrun formal derivability in the way that the Incompleteness Theorem [17] requires if we want our notion of consequence to entail all and only the truths about the integers. However, unlike definitions in terms of pure second order logic giving a definition of the numbers in terms of combinatorial necessity gives us a good starting point to explain how we do access the (necessarily incomplete) facts about numbers mathematicians are able to derive.

In particular, if the above story is right and something like H characterizes what it takes for there to be a number, then the problem of knowledge of the numbers reduces to a problem of knowledge about combinatorial possibility. Once someone knows a claim like H, they are in a position to learn any further facts about the numbers ϕ just by working out whether it is a combinatorially necessary consequence of H that ϕ[18]. Thus we get an answer to Hartry Field's question, 'why do mathematicians reliably believe truths?' and a solution to the access problem by accounting for our ability to reason correctly about combinatorial possibility.

3. Knowledge of Combinatorial Possibility

It now remains to show that there is no access problem for general reasoning about combinatorial possibility. You might well worry that facts about combinatorial possibility are just as abstract as facts about abstract objects, so reducing the mystery of access to facts about numbers to a mystery about access to facts about combinatorial possibility is not making much progress.

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[17] [Gödel, 1931]

[18] Of course, insofar as we do not have, and cannot have, exhaustive principles which axiomatize all truth-preserving reasoning about combinatorially possibility, this does not mean that human beings are in a position to learn all these further facts

But, this is not so. There seemed to be no possible way for facts about mathematical objects to ‘kick back’ and correct our methods of reasoning about these objects. However, once we turn to the case of combinatorial possibility, the situation looks different. In this section I will outline two ways for facts about combinatorial possibility to ‘kick back’ and correct our general methods of reasoning about combinatorial possibility[19]. I will then show how these two mechanisms of correction can be deployed to account for human accuracy about combinatorial possibility, and thus complete our solution to the access problem.

3.1. **Two ways for knowledge of concrete objects to correct beliefs about mathematical possibility.**

3.1.1. *Correction by* $\phi \supset \Diamond\phi$*.* First, and most obviously, there’s the inference from ϕ to $\Diamond\phi$. If you are erroneously inclined to think that a certain state of affairs is combinatorially impossible, this inclination can be corrected by learning that that state of affairs is actual. This provides a way for principles which say that too few things are combinatorially possible to be corrected. Imagine, for example, that you aren’t sure whether the state of affairs described by some mathematical hypothesis involving relations P, Q, and R is combinatorially possible. If I then point out that the relations of friendship, nephew-hood and having been in military service together apply in just this way to the royal family of Sweden, this will get you to accept that the scenario in question is, indeed, combinatorially possible.

More generally, knowledge that some scenario is physically or metaphysically possible will also allow for the inference that it is combinatorially possible, and hence yield $\Diamond\phi$ beliefs in much the same way. For, no physically or metaphysically

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[19] I stress that the mechanisms for empirical correction that I will be listing below are intended to apply to general methods of reasoning about combinatorial possibility (rather than merely to yield particular true beliefs about combinatorial possibility) for the following reason. Our knowledge of combinatorial possibility doesn’t just involve accepting the combinatorial possibility of some finite list of cases. Rather it must involve something more general. I can immediately judge that it’s combinatorially possible that there are $3,002$ cats, but it would be implausible to suppose that this is as a result of my ever seeing and counting $3,002$ of anything. Hence, if we are to appeal to correction by experience to account for the kind of positive knowledge of combinatorial possibility which humans actually have, we need to posit knowledge of particular claims about combinatorial possibility together with some kind of process of generalizing from particular experiences, and then correcting these generalizations.

possible scenario can be ruled out by the most general principles of how any objects can be related by any relations. Hence, we can reliably infer that what's physically or metaphysically possible is also combinatorially possible. So, for example, if you know that it would be physically possible to have $1,000$ different chess boards set with chess pieces, each in a different configuration, then you can reliably infer that this state of affairs must be combinatorially possible as well. In this way, the same pressures which cause us to have accurate beliefs about physical possibility can help give us accurate beliefs about combinatorial possibility.

3.1.2. *Explanatory Pressure Promotes Some Generalizations* $\neg\phi$ *to* $\neg\Diamond\phi$. Second, perhaps more surprisingly, the need to explain certain regularities in what's actual can push us toward the conclusion that certain states of affairs are combinatorially impossible.

Suppose, for example, that someone thought it was combinatorially possible for 9 items to differ from one another in which of three predicates they have (e.g., for 9 people to choose different combinations of Sundae toppings from a Sundae bar containing three toppings). This person would have to explain the striking and apparently law-like regularity that, even when we consider the most malleable of objects and most variable of relations, we never wind up observing more than 8 such objects. They would have to change their physics to explain why apparently random processes of flipping three coins never generated the forbidden 9th possible outcome. Furthermore, whatever physical explanation they chose would have to apply at every physical scale we can observe, from relationships between the tiniest particles to relationships between countries and solar systems. Also, they would have to explain why the same regularity held, in apparently exactly the same way, with regard to much less concrete subject matter like poems. Try as you may, you will never manage to think up a poem with 9 different stanzas, each of which differs from the all the others in regard to which of three poetic themes it mentions.

In theorizing about the world we can explain patterns in what actually occurs in terms of *some combination of*: a) general facts about what is combinatorially possibility b) specific metaphysical or analytic facts about the properties and relations in question and c) contingent scientific laws. In some cases considerations of theoretical elegance will push us to favor an explanation by appeal to facts about combinatorial possibility.

It might seem like there could be nothing to choose between these three different ways of accounting for the regularities I just mentioned. However, this is an illusion. Firstly, most laws don't apply to all kinds of objects, so they can't be combinatorial necessities. For example, '$\forall x \forall y \forall z$ if taller(x,y) and taller(y,z) then taller (x,z)' is an exceptionless law, but not a candidate for being true merely in virtue of its object-and-relations structure since '$\forall x \forall y \forall z$ if admires(x,y) and admires(y,z) then admires(x,z)' has the same structure but is false. Secondly, when a principle *does* apply to all kinds of objects and fits elegantly into our framework of reasoning about combinatorial possibility that elegance is a prima facia advantage.

3.2. **Three Just-So-Stories.** Now that we have these ways for facts about combinatorial possibility to 'kick back' and correct our beliefs about combinatorial possibility in mind, let us turn to the task at hand: explaining how creatures like us could have gotten correct methods of reasoning about combinatorial possibility.

Recall that the access problem is at heart an explanatory demand. Philosophers who take mathematics to describe mind independent abstract objects seem thereby committed to positing some kind of supernatural assistance shaping our mathematical intuitions[20] or a profound coincidence whereby we just randomly happened to get intuitions that correctly match the facts about an independent realm of abstract mathematical objects.

I will now attempt to defeat this impression by giving three examples of how perfectly ordinary processes could account for the (general) accuracy of our armchair reasoning about combinatorial possibility - and thus for our access to facts about mathematical objects via the story above.

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[20] e.g., something along the lines of platonic recollection or Descartes' benevolent creator shaping our mathematical intuitions

My first and simplest just-so-story appeals to conscious suggestion and correction by experience within a person's lifetime. This story relies only on humans' ability to observe objects and arrive at elegant rather than gerrymandered theories. Each person faces the task of predicting and explaining the observed behavior of the various physical objects around them. To do this they arrive, consciously or unconsciously, at various elegant generalizations and systematic ways of forming new expectations about the world.

At this point the useful idea of looking for combinatorially necessary principles - general laws governing what patterns of relationships between objects are in principle possible - may occur to them, and they may tentatively adopt some new principles for reasoning about combinatorial possibility. The general methods of reasoning which they pick out as governing combinatorial possibility will tend not to be too restrictive by virtue of the need to not rule out what's actual (as per our first mechanism of correction), and tend not to be too permissive by virtue of the need to account for regularities in what's actual (as per our second mechanism of correction).

For a concrete example of what this kind of pressure towards correct reasoning about combinatorial possibility could look like, consider the predicament of creatures with observational and linguistic abilities largely like our own, faced with the task of predicting what symbol inscriptions of various kinds they will and won't find. Like us, they can observe and produce inscriptions of strings of letters in various different media and they want to predict what patterns of letter inscriptions they will ever see written in gold, written in ink, carved in cork, stored in patterns of electricity in a computer, etc.

Now there turn out out to be many generalizations which apply to all these physically very different kinds of objects (gold leaf on top of paper, ink soaked into paper, scratches in cork, resistors in a computer in various states). For example, we won't ever see finitely many gold letters related in such a way as to satisfy the syntactic properties associated with being a proof of $0 = 1$ in Peano Arithmetic, nor will we see finitely many ink marks related in such a way..., nor finitely many scratches in cork etc.

In light of these shared laws, it will often be both more theoretically elegant and more practically efficient to explain patterns in what letter inscriptions we will see in gold, ink, etc. by a combination of general laws about what patterns of relationships between objects are in principle possible with different specific restrictions arising from physical and economic facts about each different medium, rather than by positing (and storing) separate and unrelated laws about what's physically possible by way of inscriptions in each different medium as you encounter it. Thus there is pressure to admit certain general laws about what patterns of relationships are in principle impossible.

This kind of search for explanatory generalizations can also lead one to relatively strong positive principles about what's combinatorially possible. For example, even if all the string inscriptions these creatures ever encounter are relatively short, there's pressure for them to recognize the combinatorial possibility of string inscriptions of arbitrary finite size arising from closure principles. Many closure principles which smoothly predict the facts about what short strings are physically possible will have the consequence that very long strings are combinatorially possible - even strings which are too long to physically realize given the number of fundamental particles in the universe. Take, for example, the principle that if it is combinatorially possible to have a string of letter inscriptions with a certain property, then it is combinatorially possible to have a 'doubled' string which concatenates two strings of the relevant kind, one after the other. The alternative to accepting the combinatorial possibility of such long strings would be to hold the much more complex theory where each such closure condition has an exception clause for sufficiently large instances.

Now it may seem strange to imagine broadly scientific generalizations from experience extending and correcting our reasoning about combinatorial possibility as above. Wouldn't such correction by experience have to leave traces in the phenomenology or justification of our reasoning about combinatorial possibility? Yet we don't appeal to any kind of memories of past experiences with concrete objects when reasoning about what patterns of relationships between objects are combinatorially possible.

I will argue that we actually have independent reason to believe that experience can suggest and correct mathematical beliefs without leaving any such traces in our mathematical practice. Think about the development of the kinds of hunches that guide mathematical research. Mathematicians don't choose which proof strategies to try by tossing coins, but rather on the basis of less than fully confident hunches about how certain things should turn out. They have some ideas of what proof strategies are likely to work out or not, antecedent to actually trying them. These ideas develop and improve over time. So it seems highly plausible that past research experiences are causally involved in leading them to have the right hunches. Nonetheless when they say that one strategy looks promising and another one looks unpromising, they don't consciously summon up cases of similar strategies being tried or otherwise appeal to claims about their history and past experiences for justification. If experience can correct without showing up in the justification of the relatively unconfident hunches that guide research, why shouldn't the same be true in with regard to the more confident beliefs and inferences which figure in mathematical proofs? [21]

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[21] Or consider the following, more interesting example of correction by experience leading people to accept propositions without then being disposed to cite experience in justifying those propositions: In the Monty Hall game show, there's a car behind one door, and goats behind two others. You are asked to pick a door, and then Monty opens another door to reveal a goat. Now you have the opportunity to switch doors or to stay with the one you originally chose. When presented with the Monty Hall problem, many people initially find both of the following two analyses attractive: a) switching doors doesn't help because your only relevant knowledge is that a car is behind one of the doors, so you should assign each door probability $\frac{1}{2}$ of having the car b) switching doors does help because there's a $\frac{1}{3}$rd chance that the door you first chose had the car, hence a $\frac{2}{3}$rds chance that the car is now behind the other door.

Then, the mere experience of playing around with a computer simulation, or looking at statistical descriptions of actual cases convinces them that the first analysis is the fallacious one. A recent NY Times article on cognitive illusions about the Monty Hall problem, linked to a computer simulation of the contestant's dilemma, which kept statistics for the results of switching vs. not switching in order to convince readers that its arguments for switching were correct [Tierney, 2008]

A second just so story takes our development of good general methods of reasoning about combinatorial possibility to have happened more slowly, over the course of the whole of human history rather than a single individual's lifetime. Here our reasoning about combinatorial possibility is subject to a process of generalization and correction in response to experience, much as in the story above. However, each person gets most of their principles for reasoning about combinatorial possibility by picking them up from the people around them. Very rarely someone will suggest a way of revising or extending the methods of reasoning about combinatorial possibility which are generally accepted. We can then imagine the corrective mechanisms mentioned above functioning at the level of popular adoption of such revisions causing theories that seem to elegantly predict and explain regularities in what's actual to enjoy more popularity and causing those that turn out to make false predictions to be rejected once this is discovered. In this way our general methods of reasoning about combinatorial possibility could get developed and corrected over the course of the history of ideas.

Finally, a third possible just-so story takes the methods of correction mentioned above to function on an evolutionary level. If reasoning about combinatorial possibility were 'hardwired' in some sense that precluded correction by experience, this could work in a number of different ways. There might be a special brain system dedicated to reasoning about combinatorial possibility. Or there might be a general brain process that nudges us towards the belief that certain things are physically possible, and then a conscious process of inference to the best explanation which leads us to distinguish some apparent physical laws as combinatorially necessary truths rather than mere physical laws, or metaphysical necessities of some other kind[22]. Though evolution may not care about elegance and theoretical beauty in

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Note that in these cases experience doesn't prompt the subjects to find new arguments. Rather, it changes their evaluation of the arguments they already have. Rather it leads them to believe the intuitively true claim that one of the arguments they already had ('There is a 1/3 chance the car was behind the first door I picked, hence a $\frac{2}{3}$rds chance that it is not, hence I should switch') provided sufficient justification for switching doors, by getting them to give up the other conflicting analysis which had previously seemed attractive.

[22] See Spelke's experiments with infants, e.g., [Spelke, 2009] for an example of the kind of data which might suggest that some methods of reasoning about what patterns of relationships between

quite the sense that we do, mental resources are expensive and those methods of reasoning that could be encoded in the simplest manner and handle the most general situations would be favored.

Ultimately it's an empirical question if one or some combination of these stories is right. All the realist about mathematical objects needs to do to defeat the access problem is to provide one coherent explanation for human accuracy about mathematics which is compatible with what we know now. I have attempted to provide a range of different just-so stories in order to to show that my solution doesn't depend on any particular plausibility judgement about the importance of nature, nurture or individual experience in determining mathematical beliefs. I also wanted to show that the key idea of this paper - the world corrects our general methods of reasoning about combinatorial possibility and deploying these methods of reasoning leads us to true beliefs about mathematical objects - can be realized by a number of different scientific stories about the development of human cognition.

3.3. Two Classic Challenges for Modal Epistemology.

Before concluding this section, it may be helpful to go into a bit more detail about how the stories above address two traditional challenges from the literature on modal epistemology.

One challenge for philosophers who would account for our knowledge of *metaphysical* possibility has been how to account for our ability to correctly distinguish metaphysical necessities from mere physical laws. We have a leg up on this problem when we turn our attention from metaphysical possibility to combinatorial possibility. For, as we just saw, combinatorially impossible claims are supposed to demand a pattern of relationships which it would be in principle impossible for any relationships between objects to satisfy. So these claims must remain false when you substitute one predicate or relation symbol in for another. Thus many candidate physical laws couldn't possibly be combinatorial necessities. This gives us a substantial, if not exhaustive, source of information which can guide our reasoning

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objects are combinatorially possible are relatively innate. Further results along these lines could suggest that children have good methods of reasoning about combinatorial possibility before they are in a position to do much personal experimentation with concrete objects, or hear good methods of reasoning advocated in the classroom.

about which exceptionless laws are combinatorially necessary - a source of information which has no obvious analog in the case of reasoning about metaphysical possibility.

Another challenge concerns the role of conceivability judgements in modal reasoning. For example, if you want to know whether it is possible that ϕ, you might imagine some pattern of dots connected by arrows in such a way as to satisfy ϕ. One might worry that substantial access to facts about mathematics and combinatorial possibility is already built into our predispositions to find certain situations a priori conceivable and that the workings of this faculty of conceivability are immune to correction by experience, and perhaps immune to any of the kinds of correction outlined above.

I have two responses to this objection. First, even if the kinds of mental pictures we can entertain are relatively fixed and innate, experience can still correct our conceivability judgments by correcting the way we use these mental pictures: what we take ourselves to be conceiving via entertaining a certain ghostly visual image, and generally how we read modal conclusions off of particular pictures [23]. Second, whatever aspects of our use of sensory imagination and though experiments **are** innate and immune to correction by experience can nonetheless be corrected by natural selection as discussed in our final just-so-story.

4. UNDERDETERMINATION OF THEORY BY EVIDENCE WORRIES

I will now consider a family of objections to the view just outlined. In order for an account of human access to facts about combinatorial possibility along the lines sketched above to work, we need to suppose the following: the kinds of general

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[23] To see what I mean by the claim that our use of visual imagination to judge facts about combinatorial possibility involves a substantial theoretical component, consider how you might use visual imagination to convince yourself that rotating certain match-sticks can transform one figure into another. Now contrast this with the very different ways that visual imagination can figure in your thinking about how it would be possible for Apple stocks to behave in the future. These two uses of visual imagination seem to involve very different ways of associating a picture with a state of affairs which that picture is taken to represent. What is, in some sense, the very same mental image as of red lines swerving on a white background could be used to conceive of either the possibility that a stock undergoes certain changes in value over time (via the second way of using pictures) or the possibility of piece of a paper being decorated with red and black inkmarks in a certain way (via the first).

principles for reasoning about combinatorial possibility which seem to best predict and explain the facts about what's combinatorially possible with regard to small (finite and countable) collections of concrete objects will frequently also be ones which are correct in their consequences for larger collections as well. But is it really plausible to suppose that this generalization from cases is reliable in its application to mathematics?

In the previous section I argued that it's not at all miraculous that we wound up accepting simple, seemingly universal, principles of reasoning about combinatorial possibility rather than merely a collection of case specific principles of reasoning about physical objects of particular types or objects or collections of a certain size. The worries I am now considering approach these same notions from a different perspective. If we did universalize the simple elegant principles that best explain our experience with relatively small concrete collections and use them to reason about combinatorial possibility on arbitrary collections, how miraculous would it be for this process to lead us to correct general principles of reasoning about combinatorial possibility?

This worry can be developed in a few different ways.

4.1. **Generalization from Cases is Completely Unreliable with Regard to Mathematics.**

First, one might maintain that generalization from cases is completely unreliable with regard to mathematics[24]. However, if you take this position it raises a serious problem about how to make sense of a phenomenon which we have already noticed. Working mathematicians frequently use hunches developed from past experience and judgements of general plausibility or theoretical attractiveness to guide their research. They don't pick whether to try to prove ϕ or $\neg\phi$ by tossing a coin, but rather on the basis of their feelings of theoretical plausibility and past mathematical experience. Indeed, professional mathematicians will often

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[24] Frege seems to have thought this about arithmetic [Frege, 1980] pg. 16

cite computational searches that verify a great many cases as evidence that the claim holds in generality[25].

In the section above, we considered this phenomenon as evidence that experience can correct judgements without being cited as evidence for these judgements. However, if we want to make sense of this aspect of mathematical practice we must also admit that experience can be a moderately *reliable* corrective to mathematical beliefs. Thus it can't be the case that something about the complexity or variety of mathematical objects makes the kind of elegant generalization from cases we find in the sciences utterly unreliable when applied to the mathematical realm.

4.2. **Generalization From Dealing With Small Collections Can't Account for the Degree of Mathematical Knowledge We Have.** Second, one might worry that the base of knowledge of concreta which we have is too small to support the *degree* of combinatorial (and hence mathematical) knowledge we have. One might advance the following analogy: saying that elegant generalization from knowledge of finite and countable collections yields principles which accurately describe the larger collections considered in pure mathematics is like saying that inference to the best explanation plus observations of birds in New Mexico allows us to learn about birds in Canada as well. Presumably in the ornithological case we need to go gather more data in order to get true beliefs about birds in Canada. But in the mathematical case, if my story is right, we can't gather more data. Thus facts about combinatorial possibility with respect to larger collections can appear to remain permanently a mystery.

I want to respond to this worry by accepting the analogy and claiming that it actually fits the current state of human knowledge with regard to facts about the higher infinite rather well. Even in the case of birds, we can know some things about birds in Canada just by inference to the best explanation from the facts about the birds in New Mexico. If we discovered tomorrow that some new island had never yet been visited by explorers contained birds, I think we would reasonably expect

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[25] They of course do not do this naively. If they already know that counterexamples would have to be huge they wouldn't change their judgements because no small counterexamples were found.

many facts to carry over: that any birds on that island would have DNA, that they would have hollow bones etc. Our expectations about the new island would just be very **sparse** and relatively **unconfident** relative to our beliefs about birds in locations that we have got a chance to observe.

But this is just what happens with regard to our knowledge of what's combinatorially necessary with regard to large collections[26]. Our knowledge about what is combinatorially necessary for large collections is very **sparse**: for example, the Continuum Hypothesis is a fairly simple question involving sets of (relatively) very small size, yet it is known that both the truth and the falsity of CH are compatible with all the first order combinatorial facts about the sets embodied in the standard Zermelo-Fraenkel axioms for set theory[27]. Our knowledge about what collections of objects are combinatorially possible is also relatively **unconfident**. Sociologically, mathematicians are frequently much more confident in their claims about numbers, sets of numbers and sets of sets of numbers than in the distinctive claims of set theory about what much larger patterns of mathematical objects would have to be like.

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[26] Here it may, again, it may be helpful to contrast my theory with Quine's. Although both theories take correction by experience to play an important role in accounting for mathematical knowledge, my theory and Quine's have very different consequences for what one should say about cases where mathematical facts have little or no bearing on ones expectations for experiences with concrete objects. On Quine's view we are justified in believing in whatever mathematical objects we quantify over in our best physical theories. So *absence* of relevant physical experience motivates negative claims about the height of the hierarchy of sets. Thus, insofar as our dealings with the physical world don't require us to consider higher regions of the hierarchy of sets, the Quinean picture suggests that Occam's razor should push us to reject the existence of sets at these higher levels. Thus, on this picture lack of empirical pushback on reasoning about the sets motivates definite negative conclusions about them.

In contrast, my proposal says that large sets of various kind exist if and only if this is a combinatorially necessary consequence of a certain ontologically exuberant hypothesis H describing the intended behavior of the hierarchy of sets. (Recall that the hierarchy of sets is supposed to contain, at each level, sets corresponding to every possible way or choosing from the sets below, and to contain levels which extend - in some sense - as far as possible.) But there is no intuitive pressure, analogous to Occam's razor, to say that fewer rather than more objects would be needed to satisfy the hypothesis H. Thus, on my view the divorce between certain claims about the higher infinite and experience motivates pessimism about how much we will be able to learn about the behavior of large sets, whereas on Quine's it motivates active denial that there are any such sets.

[27] The continuum hypothesis states that there are no sets whose cardinality is intermediate between the cardinality of the real numbers and that of the natural numbers. See [Jech, 1978] pg 176-186 for the proof that CH is independent of the Zermelo-Fraenkel axioms

Thus, I think this last worry points to something that is a good feature rather than a flaw of the account at hand: it explains why we have so relatively little knowledge of what's combinatorially possible with respect to large collections, and hence relatively little knowledge of the corresponding facts about higher set theory.

5. Conclusion

In this paper I have proposed a two-part strategy for solving the access problem. First we can characterize what it takes for mathematical objects like numbers and sets to exist in terms of combinatorial possibility. This reduces the problem of accounting for human access to facts about mathematical objects to a problem of accounting for human accuracy in our general methods of reasoning about combinatorial possibility.

Second, we can solve the resulting 'accuracy problem' by considering the role of reasoning about combinatorial possibility in our attempts to predict and explain the behavior of concrete objects in the world around us. I have suggested a number of ways in which the push to predict and explain the common behavior of all concrete objects by appeal to general principles about combinatorial possibility can be leveraged to explain the fact that we employ correct general principles of combinatorial possibility when reasoning about pure mathematics.

Appendix A. Formalism for Statements of Combinatorial Possibility

To give a very concrete idea of what I take the truth conditions for claims which describe what is actual in terms of what is combinatorially possible, let me introduce a little formalism.

I will use the $\Diamond$ operator to express claims about what is combinatorially possible. I will use subscripts like $\Diamond_{F,G}$ to express claims about what is combinatorially possible *given* the facts about how some predicates F, G actually apply. So, for example the claim, 'given the kittens and baskets there are, it's combinatorially impossible that each kitten slept in a distinct basket' gets written as follows:

> BASKET SHORTAGE $\neg\Diamond_{kitten,basket}$ ($\forall$x)(kitten(x)$\supset$($\exists$y)[basket(y) & slept(x,y) & ($\forall$z)(kitten(z) & slept(z,y) $\supset$ z=x)])

Now what does it take for such a claim to be true? We can mimic the my intended truth conditions for claims of this kind by thinking about a web of mathematical models ('worlds'). A starting world W_i consisting of a choice of domain and extensions that match those of the actual world. However, each starting world can share objects in its domain with a vast number other worlds W_j. These other worlds W_j have domains and extensions for predicates corresponding to each distinct combinatorially possible way of carrying out the following two tasks. First, form a domain for W_j by choosing some number, possibly 0, of objects x_k (which may be from the domain of W_i). Second, give each n-place predicate in the language an extension by choosing some collection of n-tupples from within this domain from within the domain.

Now ordinary sentences that don't contain the $\Diamond$ will be true or false at world W_i as determined by the usual Tarskian rules applied to the domain and extension of W_i. In order to characterize the truth conditions for sentences involving the diamond at W_i we must look further out to other worlds W_j. A sentence of the form $\Diamond\phi$ will be true iff there is some W_j that makes ϕ true.

A sentence of the form $\Diamond_{F,G...}\phi$ is more demanding. Intuitively the idea is that $\Diamond_{F,G}\phi$ iff there is no way of adding or removing objects from the extensions of predicates other than F and G so as to make ϕ true. More concretely a sentence of the form $\Diamond_{F,G\ldots}\phi$ is true at a world W if and only if there is some other world W' making ϕ true *while preserving the facts about how the subscripted predicates F,G... apply in W*, in the following sense:

- The domain of W' includes all the objects that are in the extension of F, G or any of the other subscripted predicates at W.
- The extension of F in W' includes all and only the objects in the extension of F in W; the extension of G in W' etc.
- ϕ is true at W'.

Accordingly we can imagine the truth value of sentences like BASKET SHORTAGE as follows.

> BASKET SHORTAGE $\neg\Diamond_{kitten,basket}$ (∀x)(kitten(x)⊃(∃y)[basket(y) & slept(x,y) & (∀z)(kitten(z) & slept(z,y) ⊃ z=x)])

First look at the actual world, and find a mathematical model which reflects it by containing a domain with as many objects as there are items in the world, and assigning one object to the extension of kitten for each kitten there is one object to the extension of basket for each basket there is etc. Now think about all the other models which preserve the extension of kitten and basket, i.e., those worlds which have all the same objects in the extension of kitten and basket as the model corresponding to the actual world does. These worlds can differ from the actual world in many ways, including assigning different things to the extension of the 'slept in' relation. The claim that $\neg\Diamond_{kitten,basket}$ ∀x (kitten(x)⊃[∃y basket(y) & slept(x,y) & (∀z kitten(z) & slept(z,y) ⊃ z=x)]) will be true iff it is not the case that one of these worlds is one in which the $\Diamond$ free sentence ∀x (kitten(x)⊃[∃y basket(y) & slept(x,y) & (∀z kitten(z) & slept(z,y) ⊃ z=x)]) comes out true.

To see how this machinery allows us to make sense of nested claims, consider the following sentence.

> POSSIBLE BASKET SHORTAGE: $\Diamond_{basket}\neg\Diamond_{kitten,basket}$ ∀x (kitten(x)⊃[∃y basket(y) & slept(x,y) & (∀z kitten(z) & slept(z,y) ⊃ z=x)]).

This says that given the number of baskets there actually are, it would be be combinatorially possible for there to be so many kittens that, not only do some kittens have to sleep in the same basket but, it would be combinatorially impossible for each kitten to sleep in a different basket.

To see whether this sentence is true, we can again start by finding a mathematical model world W_a which reflects the objects that exist and the extensions given to predicates. Now we see if we can find a world $W_{a,i}$ which preserves the extension of basket in this W_a while having the following property: there is no third world $W_{a,i,j}$ which both

- preserves the extensions of both basket and kitten in $W_{a,i}$ (and hence preserves the extension of basket in W_a)
- satisfies ∀x(kitten(x)⊃[∃y basket(y) & slept(x,y) & (∀z kitten(z) & slept(z,y) ⊃ z=x)]).

Appendix B. Characterizing the Numbers in Terms of Combinatorial Possibility

Informally we can characterize the conditions for the existence of numbers (analogous to the conditions for the existence of holes) by saying, in essence, that the numbers are well ordered by less than, extend far enough that it would be combinatorially impossible to have finitely many objects without these objects being pair-up-able with some initial segment of the numbers, and that the numbers are as few as they can be given that they satisfy this condition. I will now show that we can state this characterization purely in the language of combinatorial possibility.

I will then show that this understanding of what it takes for there to be a set is sufficient to ground the truth of all true statements of number theory by showing that it has for any statement ϕ in the language of number theory either ϕ or $\neg\phi$ is a combinatorially necessary consequence of ϕ.

B.1. **Useful Notions.** So, let begin by showing how to understand the notions of well ordering, and isomorphism and finiteness in terms of combinatorial possibility.

We can express the idea that a two-place relation R **well-orders** the Fs (i.e. objects x satisfying $F(x)$) as follows:

- R is reflexive

$$(\forall x)\,[R(x,x)]$$

- R is antisymmetric

$$(\forall x)(\forall y)\,[((R(x,y) \wedge R(y,x)) \supset x = y]$$

- R is transitive

$$(\forall x)(\forall y)(\forall z)\,[(R(x,y) \wedge R(y,z)) \supset R(x,z)]$$

- R relates the Fs so as to satisfy trichotomy

$$(\forall x)(\forall y)\,[(F(x) \wedge F(y)) \supset (R(x,y) \vee R(y,x))]$$

- it would be combinatorially impossible (fixing the facts about what actually Fs and Rs) for some of Fs to have some property G without there being some R-least F that has that property G

$$\neg\Diamond_{F,R}(\exists x)\,(F(x) \wedge G(x) \wedge \neg(\exists y)\,[F(y) \wedge G(y) \wedge (\forall z)\,(F(z) \wedge G(z) \supset R(z,y))])$$

If we have one place predicates F and G and two place predicates R and R' we can express the idea that the Fs (under R) are isomorphic to some initial segment of the Gs (under R') by talking about how it would be combinatorially possible for some new relation, envies (written $E(x,y)$), to relate the Fs and the Gs, as follows.

The Fs (under R) are isomorphic to some initial segment of the Gs (under R') iff it is combinatorially possible that envies ($E(x,y)$) relates the Fs and the Gs in a way that satisfies the following constraints:

- Each F envies some G

$$(\forall x)\,(F(x) \supset (\exists y)E(x,y))$$

- If x and y are Fs that both envy the same G then $x = y$ i.e, the envy relation pairs up Fs and Gs in a $1-1$ fashion.

$$(\forall x)(\forall y)\,(F(x) \wedge F(y) \wedge (\exists a)\,[G(a) \wedge E(x,a) \wedge E(y,a)] \supset x = y)$$

- If x envies a and y envies b (and x and y are both Fs while a and b are both Gs) then x bears R to y if and only if a bears R to b.

$$(\forall x)(\forall y)(\forall a)(\forall b)\,([F(x) \wedge F(y) \wedge G(a) \wedge G(b) \wedge E(x,a) \wedge E(y,b)] \supset [R(x,y) \iff R'(a,b)])$$

- If $G(a)$ and a is envied by some F, and b is some G which bears R' to a, then this b is envied by some F as well, i.e., envies maps to some initial segment of the Gs.

$$(\exists a)(\exists x)\,(G(a) \wedge F(x) \wedge E(x,a) \supset (\forall b)\,[G(b) \wedge R'(b,a) \supset (\exists y)\,(F(y) \wedge E(y,b))])$$

To express the claim that the Fs are **finite**, we can say that it would be combinatorially impossible for a relation R to relate the Fs to one another Hilbert's-hotel-wise – that is, to send each thing that actually Fs to a distinct other thing that actually Fs in such a way that no two Fs R the same item, and yet there is one F left over that nothing Rs.

Thus we can think of there are finitely many Fs as abbreviating the claim that $\neg\Diamond_F$

- Each F envies ($E(x,y)$) some other F.

$$(\forall x)\,(F(x) \supset (\exists y)(E(x,y)))$$

- If some Fs x and y both envy the same G then $x = y$.

$$(\forall x)(\forall y)(\exists a)\,([F(x) \wedge F(y) \wedge G(a) \wedge E(x,a) \wedge E(x,a)] \supset x = y)$$

- Some F is not envied by any other F.

$$(\exists x)(\forall y)\,([F(x) \wedge F(y)] \supset \neg E(y,x))$$

B.2. **Characterization of the numbers.** Now given this vocabulary we can now cash out the informal characterization of the numbers as follows:

(WO) The numbers are well-ordered by the '$\leq$' relation. This can be characterized directly using the account of well ordering above.

(S) There are *enough* numbers that every finite well-ordering of objects is isomorphic to some initial segment of the numbers. We can express this in terms of combinatorial possibility by saying that it would be combinatorially impossible for there to be finitely many Gs well ordered by some relation R without the Gs being 1-1 pair-up-able with the numbers in a way that respects the $\leq$ relation. That is:

> $\neg\Diamond_{Number,\leq}($ R' well-orders the Gs $\wedge$ the Gs are finite and it is not the case that the Gs (under R') are isomorphic to some initial segment of the numbers under $\leq$.

(Minimality) The numbers are *as few as can be* given that they satisfy WO and S. We can express the idea that the numbers are as few as can be by saying that, fixing the facts about what numbers there are, it would be combinatorially impossible for some objects, the Fs, to satisfy WO and S without it being combinatorially possible to pair up the numbers with some initial segment of the Fs in an order preserving way.

By similar arguments as have been used to show the categoricity of second order logic we can show that this suffices to pin down the truth conditions for ϕ. [28]

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[28] To outline how the proof works in a legible fashion, I will introduce the shorthand $\phi[F, G]$, where this abbreviates the formula you would get by starting with an expression ϕ which contains the expressions number and less than and then replacing all instances of number with some one-place predicate F and all instances of less than with some two-place predicate G. So, for example, a sentence of the form ψ[foxes, admires] says that the foxes are related by admires in the same way that the ψ claims the numbers are related by $\leq$.

Suppose for contradiction that there is some sentence ϕ such that H does not combinatorially necessitate either that ϕ or $\neg\phi$. That is, suppose it is combinatorially possible that $H \wedge \phi$, but also that $H \wedge \neg\phi$. Then it would be combinatorially possible for $H \wedge \phi$[foxes,admires] to be true as well as for $H \wedge \neg\phi$ [hounds, likes] to be true. Furthermore it would be combinatorially possible for both to be true at the same time i.e., it is combinatorially possible that $(H \wedge \phi)$[foxes,admires]$\wedge(H \wedge \neg\phi)$[hounds, likes].

However, if when we consider what is required by this supposedly combinatorially possible state of affairs, an incoherence emerges. From the fact H[foxes,admires] and H[hounds,likes] we can deduce by informal reasoning about combinatorially possibility that it would be combinatorially possible for the foxes and the hounds to be paired up 1-1 in a way that respects the behavior of

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admires and likes. This is ensured by the fact that admires is supposed to well-order the hounds and likes to well order the foxes, together with the minimality condition (3) above. That is, for all foxes a and b, a admires b if and only if a' likes b', where a' and b' are the hounds which a and b are (respectively) paired with. Now, the possibility of this kind of structure-preserving paring implies that for each sentence ψ, ψ[foxes, admires] will be true if and only if ψ[hounds,likes]. But this contradicts $(H \wedge \phi)$[foxes,admires]$\wedge(H \wedge \neg\phi)$[hounds, likes]. Thus there can be no such sentence ϕ.