Why Structure Preserving Logical Possibility?

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Abstract

Many philosophers have defended treating logical possibility as a primitive modal notion, approximately interdefinable with entailment. Recent philosophical work has employed an operator for conditional logical possibility ($\Diamond_{R_1,\ldots,R_n}$) which generalizes this notion to address epistemic access worries in mathematics and reformulate potentialist set theory. However, the notion remains underexplained and its legitimacy as a primitive concept remains contested.

This paper addresses two key challenges: (1) providing a clearer motivation and explanation for the notion of structure-preserving logical possibility, and (2) arguing that it should be treated as a fundamental modal primitive rather than reduced to set-theoretic claims.

1 Introduction

Various philosophers including Hartry Field[8] have advocated taking us to have a primitive notion of logical possibility (approximately interdefinable with entailment). The notion of conditional logical possibility generalizes this concept to capture cases like the following[1].

Consider a situation where there are three cats and two blankets. Could it be that each cat is sleeping on a different blanket? No, as per the pigeonhole principle. In this situation, there's some appeal to saying that it's logically impossible that each cat is sleeping on a different blanket. But, of course, it's not logically impossible simpliciter for each cat to have its own blanket. A scenario in which four cats are sleeping on four blankets is logically coherent. Rather, one might say that it is logically impossible for each cat to have its own blanket *given the structural facts about how cathood and blankethood apply*. So we seem to have a notion of logical possibility which doesn't just depend on general facts about logical combinatorics but also on the preservation structural features of how certain relations (e.g., cathood and blankethood) apply within a given domain.

[1] I'd like to thank REDACTED. Although mostly written before I'd found LLMs advanced enough to be useful, this paper has benefited from some nice rewording suggestions from ChatGPT.

Since this notion is distinct from plain logical possibility, we might call it conditional or structure preserving logical possibility. Recent philosophical work has used a conditional logical possibility operator $\Diamond_{R_1,\ldots,R_n}$ to do a few different jobs. For example, [2] uses it to reconceptualize the kind of knowledge of logical coherence needed for choosing acceptable mathematical posits, for the purposes of answering access worries. And [4] advocates reformulating potentialist set theory using the conditional logical possibility operator, and proposes powerful axioms for reasoning about conditional logical possibility capable of reconstructing (resulting potentailist translations of) standard ZFC set theory.

However, despite its usefulness, the notion of conditional logical possibility can seem obscure. So one might worry about: (1) whether this notion is *sufficiently clear and well-motivated* to use as a primitive in philosophical analysis, and (2) whether this notion should instead be reductively defined in terms of something else (like claims about set theoretic models). In this paper, I will try to address both worries.

In the first two-thirds of the paper, I'll try to give a clearer explanation and motivation for the conditional logical possibility than has been provided in previous work. In §2 I will review relevant background, in the form of classic motivations for accepting a logical possibility *simpliciter* operator as a modal primitive. In §3, I'll introduce the notion of key notion of logical possibility (as a generalization of logical possibility simpliciter) using a concrete example. In §3.2, I'll further explain intended truth conditions for conditional logical possibility claims by pointing out certain (systematic but limited) parallels between these claims and ones in the familiar language of set theory with ur-elements. In §4 I'll extend this story by discussing some more varied examples of conditional logical possibility claims, including ones involving nested conditional logical possibility claims.

In the final third of the paper I'll address the second challenge by discussing some positive arguments for taking conditional logical possibility as a primitive rather than understanding it in terms of set theory. In §5 I will argue that the classic motivations for accepting logical possibility simpliciter as a modal primitive generalize to conditional logical possibility, and add some further arguments for accepting the conditional logical possibility operator is a conceptual primitive, which we need not (and perhaps cannot) analyze away in terms of set theory. Finally in §6, I'll try to further motivate the latter claim by noting how using a conditional logical possibility operator lets us attractively realize a proposal of Hellman's for solving certain problems when formulating potentialist set theory.

2 Logical Possibility Simpliciter As a Modal Primitive

So let us begin with the somewhat familiar notion of logical possibility simpliciter as discussed in works like [9, 13].

In [9] Field argues that we seem to grasp logical possibility as a modal primitive, approximately interdefinable with entailment. When evaluating this kind of logical possibility $\Diamond$, we ignore all limits on the size of the universe. We consider only the broadest combinatorial constraints governing how relations can apply to objects (c.f. Frege[10]). We ignore all limits on the total size of the universe and all subject matter specific constraints on how different relations apply– considering only logical constraints that treat all n-place relations alike[2]. So for example, $\Diamond\exists x(\text{Raven}(x) \wedge \neg\text{Raven}(x))$ is false while $\Diamond\forall x(\text{Rave}n(x) \rightarrow \text{Hungry}(x))$ is true. And $\Diamond\exists x(\text{Raven}(x) \wedge \text{Vegetable}(x))$ also comes out true, even though it is metaphysically impossible for anything to be both a raven and a vegetable[3].

This notion of logical possibility is interdefinable with entailment in the following sense. Some premises entail a conclusion iff it is not logically possible for all the premises to be true while the conclusion is false[4]. And a sentence ϕ expresses something logically possible iff the falsehood of this claim is not entailed by the empty premises[5].

[2]Excepting the identity relation which I accept as logical vocabulary

[3]In terms of the correspondence between logical validity and entailment, this corresponds to the fact that we'd regard 'Bob is a raven. Therefore Bob is not a vegetable' as valid (the premises necessitate the conclusion) but not logically valid.

[4]That is, $\Gamma \vDash \phi$ iff $\neg\Diamond(\Gamma \wedge \neg\phi)$. Strictly speaking, (in cases where Γ is a finite) a collection of sentences $\gamma_1 \dots \gamma_n$, we have $\Gamma \vDash \phi$ iff $\neg\Diamond(\gamma_1 \dots \gamma_n \wedge \neg\phi)$

[5]i.e., $\Diamond\phi$ iff $\nvDash \neg\phi$

Despite close connections between logical possibility and set theory, a number of philosophers have advocated taking this notion as a primitive in philosophical analysis. At first glance, it might appear more conceptually parsimonious and/or metaphysically insightful to analyze logical possibility in terms of the existence of set models, perhaps regarding claims about logical possibility/entailment as somehow claims about set theory in disguise. After all, in many contexts we can say: some premises entail a conclusion iff it's (logically) impossible for all the premises to be true and the conclusion to be false iff there is no set model which satisfies all the premises but not the conclusion.

However Field and others [9, 7, 11, 12, 6] have powerfully replied to this common objection (as it applies to logical possibility *simpliciter*, by making an argument that we seem to grasp a modal notion of logical possibility/entailment which is distinct from, and cannot be fully analyzed away in favor of claims about set models as follows.

Any adequate way of logically regimenting and understanding logical possibility/entailment claims must vindicate the idea that what's actual is logically possible – which is clearly central to our intuitive notions of logical possibility (and validity). But if we try to identify the claim that ϕ is logically possible with the claim that ϕ is true in some set model, these assumptions looks questionable. For the actual world is strictly larger than the domain of any set model (because it contains all the sets). So it's not prima facie clear that every claim which is actually true should be true in some set theoretic model. Thus we seem to grip a notion of logical possibility on which it can be (at least momentarily) an open question whether every logically possible state of affairs has a set model.

Admittedly, we know that one can harmlessly replace claims about the logical possibility of *first order* logical state of affairs with claims about the existence of set models. But this is due to a fortunate mathematical discovery. Kriesel's squeezing argument[14] exploits the completeness of first order logic to (in effect) show that, for all such sentences ϕ, $\Diamond\phi$ iff there is a set model $M \models \phi$[6]. However, this squeezing argument only applies to claims in first order logic. And, as Boolos puts

[6] See the extended discussion of this argument in §4.3

it, “it is rather strange that appeal must apparently be made to one or another non-trivial result in order to establish what ought to be obvious: viz., that a sentence is true if it is valid”[6]. But if we identify validity with the claim that there is no set model, then we do need the above squeezing argument – or some such substantive insight – to see this.

A second attraction of understanding logical possibility and entailment via a primitive (logical) possibility operator — rather than in terms of set models — is that it lets us honor intuitions about logicality of facts about which things are logical truths. Boolos puts this intuition as follows,“one really should not lose the sense that it is somewhat peculiar that if G is a logical truth, then the statement that G is a logical truth does not count as a logical truth, but only as a set-theoretical truth.” For we have (and can prove in common systems for reasoning about logical possibility/conditional logical possibility in [4]) that $\Box G$ iff $\Box\Box G$. Hence we have a form of the following principle: it's a logical truth that G iff it's a logical truth that it's a logical truth that G.

Thirdly, accepting such a primitive logical possibility operator promises to help philosophers interested in potentialist set theory and a modal perspective on mathematics[13, 3]. For (as Parsons points out in [19]) it seems like there could be metaphysically necessary constraints on the size of the universe – especially if some kind of physicalism is true, so all objects have to ‘fit’ into space and time. Yet such metaphysically grounded limits on the size of the universe are intuitively irrelevant to set theory and potentialist claims about how hierarchies of sets can be extended, for the purposes of explicating set theory. So invoking a notion of logical possibility promises to let us assert powerful claims about the possibility (in a sense relevant to mathematics) of very large structures, while avoiding controversial and intuitively irrelevant commitments to the metaphysical possibility of such structures.

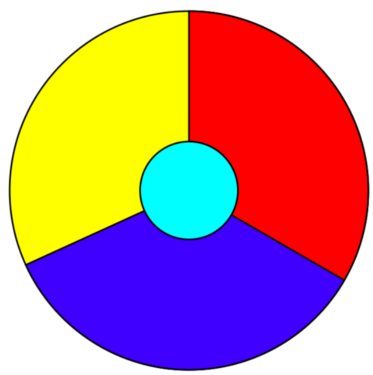
All together, I take arguments like these above to provide a decently popular and attractive case for taking a notion of logical possibility simpliciter $\Diamond$ as a primitive. But what about the newer notion of conditional logical possibility $\Diamond_{...}$?

3 From Logical to Structure-Preserving Possibility

3.1 Introducing the notion

In this section, I will try to motivate and explain the notion of conditional logical possibility $\Diamond_{\dots}$ – a generalization of the above logical possibility operator $\Diamond$, which has been argued to be useful for various purposes in the philosophy of set theory and applied mathematics[4, 2].

Consider the claim that a particular map is not three colorable. This means it is logically impossible for each tile to be colored either red, green, or blue without any two adjacent tiles sharing the same color, given the specific pattern of adjacencies in the map. This impossibility isn't due to physical constraints or practical limitations but arises purely from the structural facts about how the map-tiles are connected.



What do we mean when we think that the above map is (or isn't) three colorable? We aren't just saying that it would be practically impossible (e.g., because it is too expensive or because someone would stop you), physically impossible or even metaphysically impossible for the map to be three colored. Rather we are thinking something a bit more abstract which, for example, implies that the map can't be three-scented or three-textured either.

To motivate the idea that something like logical necessity is involved, notice that we can fully describe the map using statements that specify only two relations: 'is adjacent to' and 'is a map-tile.' From this description alone, one can logically deduce that the map cannot be three-colored. This deduction relies solely on the structural arrangement of the tiles and the general logical rules governing how properties can be distributed.[7]

For example, we might use FOL to formalize a version of the following reasoning.

> Suppose that no pair of adjacent tiles share a color. There are four tiles in the map overall. Each tile touches all three other tiles. So, no pair of tiles can share a color. All four tiles must be four distinct colors. So it's not the case that all these tiles are either red, green or blue while no adjacent tiles share a color.

Thus, we can deduce that the map is not three-colored purely through logical deductions that treat all relations of a given arity alike—just as we could similarly deduce that the map is not three-scented or three-textured.

This deduction does not depend on any physical facts (such as practical obstacles preventing three coloring) or metaphysical necessities tied to specific subject matter (like all teal things being blue or all bachelors being unmarried). It relies purely on general logical principles and the structural constraints given by the map's layout.

Rather it appears that a combination of

- *structural facts* about how the two relations in the subscript ('is a tile' and 'is adjacent to') apply
- general subject-matter neutral logical-combinatorial laws/constraints which treat all n-place relations alike[8]

[7] More specifically, the fact that the map is not three colored (i.e., the straightforward first order logical formalization of the claim that it's not the case that every tile is red, green or blue and no adjacent tiles have the same color) can be derived from true premises which (only) describe the structure formed by map-tiles in the following sense.

- These premises only employ 'is adjacent to' and 'is a map-tile' and logical vocabulary
- These premises have quantifiers which are restricted to objects which at least one of these relations applies to (i.e., those objects which are either map-tiles or adjacent to something)

[8] c.f. Frege and others on the topic neutrality of logic [17] and Bacon's more recent remark in passing, "While one can certainly have theories that include truths about, say, Socrates' life, these are not logical theories. A logic, by contrast, may include truths involving the name 'Socrates'—e.g. 'if Socrates is wise then Socrates is wise'—but if it does so it will be the general sort of truth that would apply equally to any name. Similarly, if a logic makes a claim involving the predicate 'is an electron' it is the sort of claim it will make about any other predicate: logics are not subject-specific. Logics are therefore theories closed under the rule of uniform substitution of non-logical constants" for evidence of wide uptake of this conception.[1]

guarantee that (a first order logical formalization of) the following claim obtains:

> **Not Three Colored**: It's not the case that both every map-tile is either red, green or blue and no adjacent tiles are both red, both green or both blue.

The modal operators $\Diamond_{R_1 \ldots R_n}$ and $\Box_{R_1 \ldots R_n}$ express this notion of what's logically possible, *holding fixed the structure facts* about how relevant subscripted relations $R_1 \ldots R_n$ (like 'country' and 'adjacent to in the example above) apply.

So, for example, our initial non-three colorability claim can be expressed as follows[9].

> $\neg\Diamond_{\text{adjacent,country}}$ Each country is either red, green or blue and no two adjacent countries are both red, both green or both blue.

In the example above, there's a sense in which we are only subscripting finitely much 'information'. Because the relations being subscripted happen to only apply to finitely many objects, the (actual) structure of these objects could be categorically described by some finite first order sentence A, and we would have $\Diamond_{R_1 \ldots R_n}\phi$ iff $\Diamond(A \wedge \phi)$. But note that the relevant intuitive notion of conditional logical possibility and three colorability apply just as well to cases where we have an infinite map. And an analogous strategy for replacing conditional logical possibility claims with claims about logical possibility simpliciter could not be deployed in these cases (since there might be no finite first order logical sentence A which categorically pins down the infinite structure formed by the objects under the relations $R_1 \ldots R_n$).

3.2 Parallel with set theory

While I've just tried to directly motivate and explain the acceptance of conditional logical possibility, it is also useful to explore how the intended truth values for these claims parallel certain notions from set theory. This parallel can be illustrated using the familiar framework of set theory with ur-elements, subject to some important qualifications discussed below. In particular, I will now discuss parallels using the familiar devices of set theory with ur-elements, cashing out

[9]Here I have in mind obvious FOL formalizations of the interior claims which I've written in English.

- logical possibility in terms of set models and
- structure preservation in terms of isomorphism.

For example, the claim that the map above is not three colorable is true (roughly) corresponds to the fact that no set model M can make 'The map is three colored' true while also preserving the actual structural relationships of 'country' and 'adjacent to' as they exist in the real world. In other words, no model can simultaneously satisfy the coloring conditions and maintain an isomorphism with the map's actual adjacency structure.

> $\Diamond_{\text{map region, adjacent}}$ [Every map region is either red, green, blue and no adjacent map regions have the same color].

In contrast, the claim that the above map *is* four colorable, corresponds to the following conditional logical possibility claim, which is true.

> $\Diamond_{\text{map region, adjacent}}$ [Every map region is either red, green, blue or yellow and no adjacent map regions have the same color].

Intuitively, this claim is true because there is (so to speak) a logically possible scenario which make it true that the map is four colored, but preserves the structural facts about how map-tile-hood and adjacency actually apply.

In terms of set theory and isomorphism, this corresponds to the claim that there is

- a set model M which makes 'the map is four colored' true
- a function f which witnesses the fact that M agrees with reality on the structural facts about how map-tile and adjacent to apply in the sense that
  - f is a 1-1 onto mapping from the objects related by at least one of 'map-tile' or 'adjacent to' in reality to the objects related by at least one of these relations in the model M
  - in a way that respects the relations map-tile and adjacent, i.e.,

* $\forall x \forall y[f(x) = y \rightarrow$ [x is a map-tile iff y is in the extension of 'maptile' in M)]
* $\forall x \forall x' \forall y \forall y' [f(x) = y \wedge f(x') = y' \rightarrow$[x is adjacent to x' iff $\langle y, y' \rangle$ is in the extension of 'adjacent' in M)]

For example, consider the following model M, together with a function f which maps the objects which are either map-tiles or are adjacent to something in reality to some objects in the domain of the model M in a 1-1 way as follows. f takes the central map region to 1, the top right one to 2, the bottom one to 3 and the top left one to 4. Such a combination of a model M and a function f witnesses the fact that the map above is four colorable (in the sense that it is logically possible that for all x if maptile(x) then red(x) or blue(x) or green(x) and...) .

Domain: $\{1, 2, 3, 4, 5\}$

extension for map region: $\{1, 2, 3, 4\}$

extension for red: $\{1\}$

extension for is green: $\{2\}$

extension for is blue: $\{3\}$

extension for is yellow: $\{4\}$

extension for is adjacent to:

$$\{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 1, 4\rangle, \langle 2, 1\rangle, \langle 2, 3\rangle, \langle 2, 4\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 3, 4\rangle, \langle 4, 1\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle\}$$

For the model M makes 'the map is four colored[10]' true. And the function f witnesses the fact that the model M agrees with reality in its map-tile-adjacency structure. For f 1-1 onto maps the set of objects in this structure (i.e., those related by either 'maptile' or 'adjacent to') *in reality* to the set of objects in this structure *according to model M*. And it does so in a way that respects how these two relations apply.

[10]By this, I mean the obvious first order logical formalization of the conjunction of the following two claims. For all x, if x is a map-tile then x is red, x is green or x is blue. And for all map-tiles x and y, if x and y are adjacent to each other then they aren't both red and they aren't both blue and they aren't both green.

- $\forall x$ [x is a map-tile iff f(x) is in the extension of 'map-tile' in M)]

- $\forall x \forall y$[x is adjacent to y iff $\langle f(x), f(y)\rangle$ is in the extension of 'adjacent' in M)]

So we might say f is an isomorphism between the map-tile-adjacency structure in reality and that in the model M.

This approach allows us to roughly capture the intended truth values of conditional logical possibility claims by referencing facts about set models, though this analogy has limitations, particularly concerning issues of size, which will be addressed below[11].

Note that the relevant model M and function f don't generally have to be definable[12] [13].

[11] In this sentence I am also speaking under the assumptions that traditional platonist assumptions about the existence of sets with ur-elements are correct. Obviously if there are no sets then our claims about set theory with ur-elements will not match intended truth values for claims about conditional logical possibility.

[12] When considering (conditional or unconditional) logical possibility, we are crucially considering all combinatorially possible ways that (unsubscripted) relations could apply, whether we can pick them out with some description or not. Because the specific physical map I considered happened to be finite, I could easily *describe* a model M and isomorphism f witnessing the truth of the claim that the above map could be four-colored. However the describability of the relevant witnessing model or function is inessential. Only the possibility of such a model and isomorphism matters –which will typically be witnessed by the existence of a set model and function (See the caveat below) if conventional understandings of set theory are right – regardless of whether we can concretely describe this model in our language.

For example, some conditional logical possibility claims will fix structural facts about how relations with infinite extensions apply (e.g., considering what's logically possible given the behavior of an infinite star system considered under certain relations), or assert logical possibility of claims whose truth requires the existence of infinitely many objects. In such cases, we might not be able to concretely describe any model and function witnessing the truth of a true conditional logical possibility claim (i.e. one of the form $\Diamond_{R_1...R_n}\phi$). But that makes no difference to the intelligibility of talk about what's logically possible given the structure of this infinite physical map or star-system etc.

[13] We can also state (and prove) a modal version of the famous four coloring theorem, saying that it's logically necessary that each map is four colorable (provided this map satisfying certain algebraic assumptions which plausibly all spatially possible maps must satisfy). Using the understanding of nested logical possibility claims to be explained below, we can write the following claim.

$\Box$ [The countries under adjacency satisfy the conditions for being a loopless planar graph $\rightarrow$ $\Diamond_{country,adjacent}$ (Every country is either red, yellow, green or blue and no adjacent countries are both red, both yellow both green or both blue)]

This says that it's logically necessary that it's logically necessary that (if the countries under adjacency satisfy the conditions for being a loopless planar graph) country and adjacent apply in a way that makes it logically possible (fixing the structural facts about how these terms apply) that the map is four colored.

And we can (in principle) prove this claim by creating a version of the original computer generated proof using the inference system for reasoning about conditional logical possibility advocated in [4].

4 More examples of Simple Conditional Logical Possibility Claims

I will now try to further explain and motivate the basic notion of conditional logical possibility by discussing some more diverse examples of conditional logical possibility claims. Consider the following sentence (which will be helpful for setting up our discussion of nested conditional logical possibility statements below).

Kittens: $\neg\Diamond_{kitten,blanket}$ Each kitten is sleeping on a different blanket

This says that it's logically impossible, given (the structural facts about) what kittens and blankets there are that every kitten is sleeping on a distinct blanket.

This claim will be true if there are 11 cats and 10 blankets. For, any logically possible scenario which preserves the kitten-blanket structure of reality must (in this case) also be one where there are 11 cats and 10 blankets. And (as per the pigeonhole principle) there's no possible way of choosing how the 'sleeps on' relation applies which lets it pair up 10 things with 11 things one to one[14]

On the other hand, suppose that there are actually two kittens and three blankets. Then the above claim will be false. For nothing in the structure of how 'kitten' and 'blanket' apply (i.e., the fact that they apply to distinct collections of two things and three things respectively) makes it combintorially impossible for 'is sleeping on' to pair each kitten up with a distinct blanket in so as to make'each kitten is sleeping on a distinct blanket' true. There are logically possible scenarios which preserve the actual kitten-blanket structure (by having two kittens and three blankets), and have 'is sleeping on' apply so as to make it true that each kitten is sleeping on a different blanket. We are no longer blocked from doing this by combinatorics/the pigeonhole principle[15].

[14]In terms of set models, this corresponds to the fact that no set model M makes 'each kitten is sleeping on a different blanket' come out true while preserving structural facts about how kitten and blanket would apply in this scenario (by interpreting these relations to apply in a way that's isomorphic to reality – which in this case just requires interpreting 'kitten' and 'blanket' as applying to disjoint collections of 11 and 10 things respectively),

[15]In terms of set theory with ur-elements, the truth of Kittens corresponds to the fact that there's a model M which agrees with reality on the kitten, blanket structure (as witnessed by some isomorphism f), and makes 'each kitten is sleeping on a distinct blanket' true. For example, here is one such witnessing M and f.

Model M

Here are some further examples natural language statements that can be naturally and helpfully interpreted as conditional logical possibility claims (i.e., claims about what's logically possible *holding fixed* structural facts about how some relations apply) and stated using the the conditional (structure preserving) logical possibility operator $\Diamond_{...}$.

- There's no possible way (given the structural facts about how land regions are connected by bridges in Köningsburg) for there to be a series of conected walk stages which cross each Köningsburg bridge exactly once.
- For every way of stationing defending troops in this map region in this board game, there's a way of stationing attacking troups which.../There's no possible way of adding ziplines to the Köningsburg islands and bridges such that....

4.1 Numbers

A particularly interesting and philosophically useful example of a notion expressible using (non-nested) conditional logical possibility claims is the induction axiom.

We can express the second-order induction axiom, which captures a categorical conception of the natural numbers, using the framework of conditional logical possibility as follows:[16].

Domain $\{1, 2, 3, 4, 5\}$
kitten: 1,2
blanket: 3,4,5
is sleeping on: $\langle 1, 3\rangle, \langle 2, 5\rangle$
Function f f that maps the kittens to the numbers 1 and 2, and the blankets to the numbers 3, 4, and 5.

Clearly M makes 'Each kitten is sleeping on a distinct blanket' true. And f is an isomorphic map from the cat-blanket structure in reality to the cat-blanket structure in M. That is, f is a 1-1 onto map from the set of objects which the relations 'cat' and 'blanket' apply to in reality, to the set of objects those relations relate in M, which respects the application of both relations.

[16]Famously, one can't uniquely pin down the intended natural number structure in the language of first order logic. In particular, every consistent true first order theory about the numbers which extends Peano Arithmetic will be satisfied by both the intended model and nonstandard models which combine a standard initial segment with extra numbers. Such models contain, so to speak, a copy of the natural numbers together with some copies of the integers stuck on at the end 0, 1, 2, 3.. -2*, -1*, 0*, 1*..) in such a way as to satisfy all the axioms in our first order theory.

The Second Order induction axiom $Induct_2$ below rules out such nonstandard models, by using quantification over second order objects — which are taken to witnessing all possible ways of choosing from the objects of first order quantification (in this case the numbers), whether describable in some language or not.

Combining this axiom with PA- (a combination of all the first order Peano Axioms except for the infinitely many of

Induct$_2 (\forall X) [(X(0) \wedge (\forall n) (X(n) \rightarrow X(n+1))) \rightarrow (\forall n)(X(n))]$

We can reformulate this claim using conditional logical possibility as follows[17].

- **Induct**$_\Diamond$: '$\Box_{\mathbb{N},S}$ If 0 is happy and the successor of every happy number is happy then every number is happy.

Put simply, this expresses the logical necessity that, given the structure of the natural numbers and the successor relation, if 0 is happy and the successor of every happy number is also happy, then all numbers must be happy.

This claim holds because any logically possible scenario that preserves the structural facts about how 'number' and 'successor' apply must be one where the numbers continue to form a genuine ω-sequence. So any such scenario where the numbers are as few as can be while being closed under successor (in the sense this is true of the actual numbers). So, unlike when considering nonstandard models of PA, it will be impossible for 'happy' to apply to 0 and the successor of every number but fail to apply to all numbers, in this logically possible scenario. And of course, the same reasoning applies to every other predicate we could have chosen instead of green (as can be proved in the formal system proposed in [4]). The work in pinning down an intended natural number structure is all done by box[18].

Stepping back, we can see that $Induct_\Diamond$ and the second order induction axiom $Induct_2$ both imply the same intuitive constraint on the natural number structure – one which we might equally express by talking about 'all possible ways of choosing' some numbers. $Induct_2$ assures us that there's no possible way of choosing some numbers xx which is counterinductive (i.e., xx contains the first number and the successor of all numbers it contains but not all numbers). It does this

instances of the induction schema), allows us to uniquely pin down an intended natural number structure.

[17] I write '0' below for readability, but recall that one can contextually define away all uses of 0 in a familiar Russellian fashion in terms of only relational vocabulary

[18] In terms of set theory with ur-elements, the truth of Induct$_\Diamond$ corresponds to the fact that there is no set model M and function f, such that M makes 'If 0 is happy and the successor of every happy number is happy then all numbers are happy' false, and f isomorphically maps the genuine natural number-successor structure in reality to natural number-successor structure in M.That is, f 1-1 onto maps the genuine natural numbers and objects related by successor to the objects related by these relations in M, in way that respects successor).

by quantifying over sets/second order objects (via the presumption that second order objects exist corresponding to all possible ways of choosing some numbers). $Induct_{\Diamond}$ asserts the same idea about the impossibility of choosing some numbers xx which are counterinductive, via talking about via freezing the actual number-successor structure and the presumption that conditional logical possibility facts reflect all possible ways of choosing an extension for a non-subscripted predicates.

4.2 Nested conditional logical possibility

Finally (as foreshadowed by the discussion of Boolos above), we can also nest claims about conditional logical possibility.

The meaning of nested conditional logical possibility claims can be understood by naturally extending the interpretation we used for non-nested conditional logical possibility statements.

- Above we (in effect) said that $\Diamond_{R_1 \ldots R_n}\phi$ is true at the actual world iff some logically possible scenario which **agrees with the actual world** on the structural facts about how $R_1 \ldots R_n$ apply makes ϕ true.
- So the obvious generalization would be to say that $\Diamond_{R_1 \ldots R_n}\phi$ is true relative to an arbitrary logically possible situation w, iff a scenario w' which **agrees with w** on the structural facts about how $R_1 \ldots R_n$ apply can make ϕ true.

Thus we can unpack the meaning of nested claims like the one below, as follows.

Possible Kittens $\Diamond\Box_{cat,blanket}$[If every kitten is sleeping on a blanket, at least two cats are sleeping on the same blanket]

The above sentence says that it's logically possible for kittenhood and blankethood to apply in a way that makes our earlier statement Kittens true. That is, it's logically possible for kitten and blanket to apply in a way that makes it logically necessary (given how 'kitten' and 'blanket' apply *in this hypothetical scenario*) that if every kitten is sleeping on a blanket, at least two cats are sleeping on the same blanket.

And we can see that Possible Kittens expresses a (metaphysically and logically necessary) truth for the following reason. It's logically possible for there to be three cats and two blankets. But in any such logically possible situation, it would be logically necessary (given how 'cat' and 'blanket' apply *in this hypothetical scenario*) that if each kitten is sleeping on a blanket then two cats are sharing a blanket. So in the situation in question would be one where ' $\Box_{cat,blanket}$[If every kitten is sleeping on a blanket, at least two cats are sleeping on the same blanket]' is true. Thus it's logically possible that it's logically necessary (given structural facts about how cathood and blankethood apply) that if every kitten is sleeping on a blanket, at least two cats are sleeping on the same blanket[19]

And one can state a general strategy for set theoretically paraphrasing claims conditional logical possibility (including claims that involve nested conditional logical possibility), as is done in [4][20]

[19]In terms of our parallel with set theory with ur-elements, this corresponds to the following claim.

There's a model M in which there are three things in the extension of 'kitten' and two in the extension of 'blanket'. And at this model M, the interior claim that $\neg\Diamond_{cat,blanket}$[Each kitten is sleeping on a different blanket] is true. For there's no model M' which makes 'each kitten is sleeping on a different blanket' true while preserving the the structural facts about how 'cat' and 'blanket' apply at M (in the sense that some function f isomorphically maps the cat-blanket structure in M to the cat-blanket structure at M').

For, by fact that M has 3 things in the extension of 'cat' and 2 in that of 'blanket' and f is a 1-1 map which respects how 'cat' and 'blanket' apply, M' must also have (so to speak) 3 things in the extension of kitten and 2 in the extension of 'blanket'. So, by the pigionhole principle, however M' assigns an extension to 'is sleeping on' it cannot make 'every kitten is sleeping on a distinct blanket' true.

[20]That says that:

> **Set Theoretic Translation**: A sentence $\psi = \Diamond_{R_1 \dots R_n}\phi$ is true iff there is some model $\mathcal{M}$ and function f such that and $\mathcal{M}' \models \phi$ and f isomorphically maps the set of objects related by $R_1 \dots R_n$ to the set of objects belonging to some ntupples in the extension of $R_1 \dots R_n$ in $\mathcal{M}$, in a way that respects relations $R_1 \dots R_n$.
>
> A formula ψ is true relative to a model $\mathcal{M}$ ($\mathcal{M} \models \psi$) and an assignment ρ which takes the free variables in ψ to elements in the domain of $\mathcal{M}$[21] just if:
>
> - $\psi = R^k_n(x_1 \dots x_k)$ and $\mathcal{M} \models R^k_n(\rho(x_1), \dots, \rho(x_k))$.
> - $\psi = x = y$ and $\rho(x) = \rho(y)$.
> - $\psi = \neg\phi$ and ϕ is not true relative to $\mathcal{M}, \rho$.
> - $\psi = \phi \wedge \psi$ and both ϕ and ψ are true relative to $\mathcal{M}, \rho$.
> - $\psi = \phi \vee \psi$ and either ϕ or ψ are true relative to $\mathcal{M}, \rho$.
> - $\psi = \exists x\phi(x)$ and there is an assignment ρ' which extends ρ by assigning a value to an additional variable v not in ϕ and $\phi[x/v]$ is true relative to $\mathcal{M}, \rho'$[22].
> - $\psi = \Diamond_{R_1 \dots R_n}\phi$ and there is another model $\mathcal{M}'$ and a function f which isomorphically maps the objects related by relations $R_1 \dots R_n$ in $\mathcal{M}$ to those related by these relations in $\mathcal{M}' \models \phi$.[23][24]

4.3 Caveats and Clarifications

In the previous subsections, I used parallels with set theory to clarify the notion of conditional logical possibility. However, these parallels should not be mistaken for a reductive analysis. Ultimately, I advocate for taking conditional logical possibility as a primitive. This does not mean it cannot be explained or motivated—many fundamental logical notions (e.g., quantification) were introduced before being treated as primitives.

One might ask whether conditional logical possibility can be reduced to claims about set models. While set-theoretic analogies can help articulate the concept, there are good reasons to reject this reduction. Kreisel's squeezing argument highlights a general limitation: logical necessity is constrained both by provability and model-theoretic entailment, but these constraints do not always align. In first-order logic, completeness ensures that provability and model-theoretic necessity coincide, but in more expressive languages (such as second-order logic), they may diverge.

A similar issue arises for conditional logical possibility. The fact that a sentence has no countermodels does not necessarily mean it is logically necessary in the relevant sense. If we define logical possibility in terms of the existence of set models, we risk failing to capture the full modal structure of possibility. For example, some logical necessities may not be witnessed by any set-sized model, particularly in cases involving proper-class-sized structures.

Thus, while set-theoretic formulations can approximate conditional logical possibility, they do not fully capture its nature. Taking it as a primitive aligns with the broader motivation for treating logical possibility itself as primitive, avoiding unnecessary theoretical commitments and preserving a more direct grasp of possibility.

In the two sections above I mentioned various systematic parallels between claims about conditional logical possibility and claims about set theory with ur-elements. However we should note two caveats.

5 Conditional Logical Possibility As Prior to Set Theory

With the above clarifications and motivations for accepting conditional logical possibility talk as sufficiently clear and meaningful to be treated as a modal primitive in mind, I will now turn to addressing the second worry mentioned in the introduction to this paper.

This second worry says the following. Even if you accept conditional logical talk as clear and meaningful, why complicate our fundamental ideology by accepting $\Diamond_{...}$ as a primitive logical operator? If the above claims about a close relationship between conditional logical possibility and set theory are correct, why not just cash out conditional logical possibility claims using set theory[25]?

In this section, I will develop some arguments for accepting conditional logical possibility as a primitive (rather than rejecting the notion or analyzing conditional logical possibility in terms of the claims about set models above).

I will begin by proposing two ways in which we seem to grasp *something like* notion of conditional logical possibility prior to understandings of set theory[26]

First, we often/naturally explain the intended structure of the iterative hierarchy of sets V by appeal to something like a notion of conditional logical possibility (or at least 'all possible ways of choosing'). Following Boolos[5] one can describe the intended structure of the iterative hierarchy of sets in the following way, which I will suggest can be seen as making implicit appeal to a notion like conditional logical possibility:

Height The hierarchy of sets is built up in a well ordered sequence of layers.

Width At each layer α, there are sets corresponding to all possible ways of choosing (often expressed in terms of the power set operator) some of the sets available at levels below α.

[25] I won't say much about the option of taking entailment rather than logical possibility as a modal primitive in this paper. Using logical possibility is more familiar and convenient, but I suspect little hangs on this choice.

[26] I don't claim the relevant modal notion (sometimes expressed by talk of 'all possible ways of choosing') is exactly the concept of conditional logical possibility discussed above. For example, this informal modal notion may be ambiguous or undefined in places where conditional logical possibility. I'm just suggesting the relevant pre-theoretic notion can be attractively carnapianly explicated in terms of conditional logical possibility – and is close enough to support the idea that one can grasp conditional logical possibility directly without appeal to a prior understanding of set theory.

Both elements of this picture of the intended hierarchy of sets appeal to first order logic transcending notions of something like the notion of conditional logical possibility I have advocated.

For we can explicate both of them as describing the actual structure of the hierarchy of sets by saying that this structure makes certain things conditionally logically impossible.

We can cash out [Width] as saying the following

It is logically necessary, given the set and element facts, that, for every set available at some ordinal level α, there is a set at level α whose elements are exactly the happy sets at layers below α[27].

> $\Box_{set,element}$ For each ordinal α, there's a a set whose elements are exactly the happy sets at layers below α.

And similarly the claim [Height] that the layers in the hierarchy of sets seem to be well ordered amounts to saying that there's no possible way of choosing some layer without one of the layers you've chosen being less than all the others (together with the first order logically formalizable claim that the layers are linearly ordered by 'above')[28].

Thus, we appear to grasp a notion akin to conditional logical possibility independently of set theory—one that we frequently invoke when explaining the iterative hierarchy of sets. We expect facts about set theory to generally reflect and witness facts about conditional logical possibility in the way described above[29].

Second, the expectation that set theory aligns with modal facts about something like 'all possible ways of choosing' (which implies counterfactual supporting constraints on how non-

[27] Here talk about availability layers can be cashed out in the language of set theory via talk about ordinals.

[28] A version of this point was already made in [4]

[29] C.f. Scambler making a related point in [20], "the basic intuition behind Hume's principle is at least arguably a modal one: namely, that it is possible to assign every B at least one A (distinct from those assigned to other Bs) exactly when there are at least as many As as Bs, in at least some reasonable sense of 'at least as many'. Now, assuming all possible functions actually exist, the circumstance that such an assignment is possible coincides exactly with the circumstance that there exists a function defined on A with all Bs in its range, and so Hume's principle as just stated is justifiable. In modal set theory however one is forced to countenance the idea that there is in fact no function on A with all Bs in its range, but that one is nevertheless possible. In such a case, how should we rule on the relative sizes of A and B? Hume's principle as stated tells us that there are more Bs than As, since there is no function on A with all Bs in its range. But the intuition supporting Hume's principle would seem to suggest the opposite, since the possible function corresponds to a possible assignment of the relevant kind."

matheamtical properties can apply to non-mathematical objects) as per the analysis above, seems to play an important role in our expectations about how facts about set theory can be applied.

For conditional logical necessity claims, like the non-three coloring claim §3, intuitively have power to explain why relevant first order non-mathematical claims don't obtain and could not have easily obtained. They can figure in seemingly cogent counterfactual supporting explanatory hypotheses like the following.

- The reason why some physical map has never been three colored is that it's not three colorable (in the logico-combinatorial sense evoked above).
- The reason why bus service has continually been so bad in a certain area, is that there's no set of ordered pairs (and hence no possible way of choosing how to have routes to connect) locations in town, which secures some property.
- The reason why some portion of a videogame map has seldom held by the same team for more than 50 rounds is that, for every way of stationing attacking troops, there's a way of stationing defending troops with such-and-such property.
- The reason why all instances of the induction schema are true (in our current language and all extensions of it) is that the structure of the natural numbers under succesor makes it impossible to chose some numbers that include 0 and the successor of every number it includes. Hence we can know a priori that no one will introduce a predicate (or definition with parameters), which violates induction, and we can accept all instances of the induction schema open endedly[30].

If we rejected the above traditionally expected connection between set theory and (conditional-logical-necessity-like) modal constraints on how non-matheamtical properties apply to non-mathematical objects, such explanations would look mysterious. Why should the mere fact that there happens to be no *set* that codes a three coloring function imply or explain the fact that a map has never been

[30]c.f. [18] on the notion of open endedness I have in mind.

three colored, if we renounce the assumption that sets are supposed to exist witnessing ‘all possible ways of choosing’ from objects at lower layers (as per the modal descriptions of the iterative hierarchy above)?

A third motivation[31] can be given to readers who accept the classic argument for accepting logical possibility simpliciter as a plausible modal primitive (rather than something to be reduced to set theory) reviewed in §2.That argument said that intuitively what’s actual must be logically possible, but this is not obviously true if we require logical possibility to be witnessed by the existence of a set sized model. This point generalizes exactly to the case of conditional logical possibility, as explored in the discussion of Kreisel’s squeezing argument in §2 above. So if you already accept logical possibility as a modal primitive, and accept the conditional logical possibility as a meaningful, regarding the latter as a modal primitive seems natural (since then you can say that logical possibility facts are just conditional logical possibility facts which hold fixed the empty list of relations).

So overall, I think there’s good motivation for accepting a notion of conditional logical possibility as a legitimate modal primitive which need not be analyzed in terms of existence claims about set models.[32].

[31]C.f. the points about existence claims involving actualist set models being an imperfect guide to conditional logical possibility in §3 above.

[32]Note that simply using a plain logical possibility operator doesn’t allow us to make claims about all possible ways of choosing from a given structure. So taking that step alone doesn’t let us recapture the goodness of the arguments above.

In contrast, adding a primitive second order logical quantifier would let us capture the notion of ‘all possible ways of choosing’ –and hence make sense of some of the physical explanations above. And you could capture even more such explanations by combining second order quantification and logical possibility simpliciter operator (and allowing quantifying in to the diamond of logical possibility and second order quantification) as per Hellman’s formulations of potentialist set theory. This would let you mirror explanations which are easy to give using conditional logical possibility, by saying things like: it’s logically possible to add tiles to this map in such a way that the resulting map isn’t three colorable.

However, I think there’s a kind of inelegance and (in a sense) redundancy to adopting second order quantification and a logical possibility simpliciter operator as primitives. For we seem to consider the same notion of all possible ways some predicates and relations could apply (whether independently describable or not) when grasping second order quantification and logical possibility.

6 Enhancing Logicist approaches to Potentialist Set Theory

Finally let me end by arguing that accepting the notion of conditional logical possibility above lets one attractively implement some independent suggestions made by Hellman about how to solve a certain puzzle raised by Linnebo, in the recent literature on potentialist set theory.

Potentailist set theory aims to banish puzzles about the intended height of the hierarchy of sets by explicating ordinary set theoretic claims as something like claims about how (patterns of objects with the structure of) intended width hierarchies of sets could be extended.

But a debate has arisen between two different schools of potentialism over how to implement this idea. What Linnebo calls *Putnamian* approaches translate set theory as claims about what's (metaphysically or logically) possible for all structures of objects satisfying certain axioms. In contrast, *Parsonian* potentailists translate set theory with claims about what *sets* their could be (with 'could' expressing a special notion of interpretational possibility, such that it would be interpretaitonally - but not metaphysically- possible for different pure sets to exist).

I take the allure of the Putnamian approach – if we can get away with it – to be clear. If one can formulate set theory using only independently motivated notions of metaphysical or logical possibility (rather than introducing this new and somewhat unintuitive notion of interpretational possibility), considerations of conceptual parsimony and elegance support doing so. But in [16] Linnebo (one of the most prominant Parsonian potentialists) raises the following challenge for Putnamians.

(Extant minimalist potentialist) paraphrases of basic set theoretic axioms like powerset wind up implying things like the following de re extendability principle. Any collection of objects forming an iterative hierarchy structure could be extended, existing alongside additional objects as needed to form an extended hierarchy of sets structure.

But it is not clear that such supplementation is possible for all objects. For example, if by 'possible' we mean metaphysically possible, couldn't there be *metaphysically* shy objects, which could not exist in a larger universe [33]? Couldn't there be objects which wouldn't survive the

[33]Linnebo also raises an analogous worry about incompossibles, as some potentialists paraphrases wind up requiring

addition of many more objects (as needed to form an extending hierarchy of sets structure), just as food deserts wouldn't survive the opening of a produce-heavy grocery store nearby?

If so, our justification for even the most basic set-theoretic principles would rest on false assumptions about metaphysical possibility. We could instead appeal to logical possibility simpliciter. But the intuitive concept of logical possibility (associated with validity) above, doesn't have much to say on what's de re logically possible. Also there are general controversies about how to handle quantifying in in modal logic.

Linnebo notes that one possible strategy for answering this challenge – suggested by Hellman in conversation – would be to formulate potentialist paraphrases using claims about logical/metaphysical possibility which preserve structural facts (rather than the existence of particular objects), as follows.

> A more promising option, suggested to me by Hellman, is to relax the extendability principle such that it only makes demands 'up to isomorphism' : 'Necessarily, for any model M, possibly there is a model M' which is isomorphic to M and which possibly has a proper extension. While this is promising, we need to be shown how the modal structuralist has the resources to formulate the transworld isomorphism claim

And I claim that formulations of set theory using the notion of conditional logical possibility advocated in this paper (as per [4][34]) let us implement exactly (or almost exactly) this response. For the conditional logical possibility operator lets us talk about what's logically possible given *the structure* of some how objects S are related by some relation E ($\Diamond_{S,E}$). So they let us say that te iterative hierarchy structure formed by objects under relations S,E could (logically possibly) exist within a larger extending iterative hierarchy structure of objects under relations S'E', without committing ourselves to any de re claims that specific objects related by S and E could have existed within a larger universe.

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that any two possible sets/objects playing the set roles, could exist together.

[34] See [4] for significantly more detail on how to formulate potentialist set theory using the conditional logical possibility operator (it turns out to be possible to drop second order quantification, plural quantification and quantifying in).

7 Conclusion

In this paper, I've tried to explain the meaning and usefulness of the notion of structure preserving logical possibility ($\Diamond_{...}$) in a more intuitively appealing and broadly accessible way. I've also tried to expand and supplement existing arguments for taking conditional logical possibility as a primitive.

A Lewis' Anti-Hacceitism

Perhaps one can deepen the appeal of having such a tool for sidestepping de re modality (where desired), by considering the appeal of David Lewis' anti-haccaeitism.

In [15] Lewis advocates an anti-hacceitist view on which "all contingent truth supervenes on the pattern of coinstantiation" of properties and relations, as follows.

> [W]e may be certain a priori that any contingent truth whatever is made true, somehow, by the pattern of instantiation of fundamental properties and relations by particular things. In Bigelow's phrase, truth is supervenient on being ...If two possible worlds are discernible in any way at all, it must be because they differ in what things there are in them, or in how those things are. And 'how things are' is fully given by the fundamental, perfectly natural, properties and relations that those things instantiate... As an anti-hacciatist, I myself would drop the 'what things there are' clause; I claim that all contingent truth supervenes just on the pattern of coinstantiation, never mind which particular hooks the properties and relations are hanging on. (On my view, the hooks are never identical from one world to another, but that by itself doesn't make the world discernable.)[15]

So, interestingly, the kinds of fundamental facts Lewis allows to distinguish metaphysically possible worlds (i.e., facts about the "pattern of coinstantiation" for properties and relations, as opposed to haccaetistic facts about which objects have a given property) are exactly the kinds of structural facts which the conditional logical/metaphysical possibility operator preserves. One can understand an assess conditional logical/metaphysical possibility claims by considering only the kinds of uncontroversial features of metaphysically possible scenarios that distinguish 'discernable' possible worlds (i.e., the pattern of how properties and relations corresponding to the list of subscripted atomic relations names $R_1 \ldots R_n$ are coinstantiated). There is no need to presume controversial extra structure like haccieties (which Lewis rejects) or a counterpart relation (which Lewis regards as non-fundamental).

In this way, claims about structure preserving possibility only attempt to preserve a very clear and uncontroversial (and plausibly fundamental) features of logically/metaphysically possible scenarios: what Lewis calls a pattern of co-instantiation. Accordingly conditional possibility claims (unlike de re possibility claims sometimes used to formulate potentialist set theory) can be understood via fairly direct appeal to features everyone agrees that all logically/metaphysically possible scenarios have – namely a pattern of how properties and relations are (co)instantiated.

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